Operator Complexity Growth at Short and Long Times

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Based on work with Eliezer Rabinovici, José Barbón & Ritam Sinha

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- Given such a definition, how does this complexity evolve in time?

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Idea:

Use the number of commutations with Hamiltonian to asses complexity of $\mathcal{O}(t)$.

Example in SYK

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- How fast is the operator growing? [see Roberts, Stanford & Streicher]
- What happens when the operator reaches the system size?

Quantitative Definition of Operator Complexity

Based on Parker, Cao, Avdoshkin, Scaffidi & Altman
arXiv:1812.08657

Idea:
Use the set of operators
$\{\mathcal{O},\, [H,\mathcal{O}],\, [H,[H,\mathcal{O}]],\, [H,[H,[H,\mathcal{O}]]],\, \ldots\}$ to form an orthonormal basis in the space of operators. In some sense, each operator is more complex than the other.

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Define the inner product on the space of operators

$$(\mathcal{O}_1, \mathcal{O}_2) \equiv \frac{\mathrm{Tr}(\mathcal{O}_1^\dagger \mathcal{O}_2)}{\mathrm{Tr}(\mathbb{1})}, \quad \mathrm{Tr}(\mathbb{1}) = \mathcal{N}$$

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$$\begin{array}{cccc} \mathcal{O}_0 & \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_3 \\ \mathcal{O}_0 & \frac{1}{b_1}[H, \mathcal{O}_0] & \frac{1}{b_1 b_2}([H,[H,\mathcal{O}_0]] + b_1^2\mathcal{O}_0) & \frac{1}{b_1 b_2 b_3}[H,[H,[H,\mathcal{O}_0]]] + \dots \end{array}$$

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For a generic Hamiltonian we expect this basis to span the Hilbert space of operators and therefore $b_n = 0$ only when $n = \mathcal{N}^2$.

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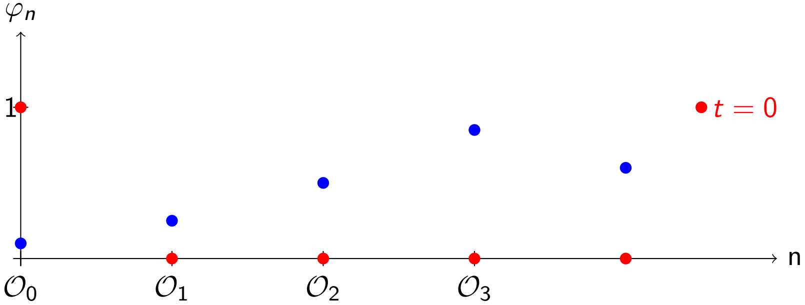

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$$\boxed{\dot{\varphi}_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)}$$

with initial and boundary conditions

$$\varphi_0(0) = 1, \varphi_{n>0}(0) = 0, \varphi_{-1}(t) = 0.$$

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How does C_K evolve in time?

The answer depends on $\varphi_n(t)$ whose dynamics depend solely on the Lanczos coefficients $\{b_n\}$.

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$$b_n \sim \alpha n$$

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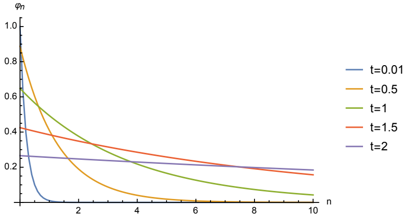

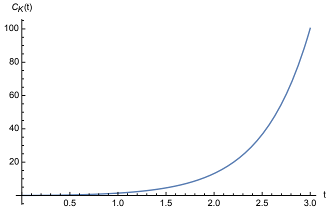

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Can $C_K(t)$ keep on growing exponentially forever?
We think that in a finite system, it cannot.

To show this, let's look at the *moments* μ_{2n}, of the autocorrelation function

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If we expand the operator in the energy basis,

$$\mathcal{O}_0 = \sum_{a,b=0}^{\mathcal{N}} O_{ab} |E_a\rangle\langle E_b|$$

the moments have the simple form

$$\mu_{2n} = \frac{1}{\mathcal{N}} \sum_{a,b=0}^{\mathcal{N}} (E_a - E_b)^{2n} |O_{ab}|^2$$

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How can we use this to constrain the b_n-sequence?

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μ_{2n} are given by the weighted Catalan numbers which are sums over *Dyck paths* of length $2n$

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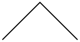
$$\mu_4 = b_1^4 + b_1^2 b_2^2 =$$  $+$ 

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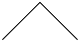
$\mu_4 = b_1^4 + b_1^2 b_2^2 =$  $+$ 

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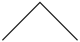
$$\mu_4 = b_1^4 + b_1^2 b_2^2 =$$  $+$ 

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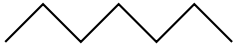
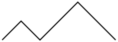
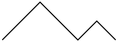
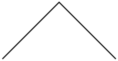
$=$  $+$  $+$  $+$  $+$ 

μ's and b's

μ_{2n} are given by the weighted Catalan numbers which are sums over *Dyck paths* of length $2n$

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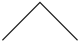
$$\mu_4 = b_1^4 + b_1^2 b_2^2 =$$  $+$ 

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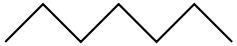
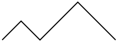
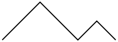
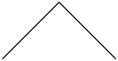
$=$  $+$  $+$  $+$  $+$ 

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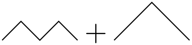

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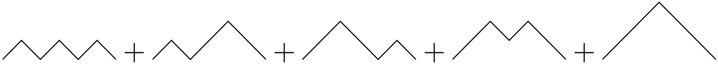

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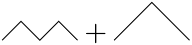
$$\mu_4 = b_1^4 + b_1^2 b_2^2 =$$ 

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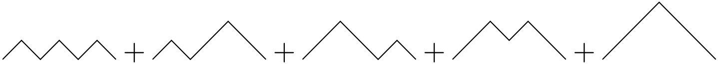
$$=$$ 

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therefore for non-decreasing b_n-sequences

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$$C_n \sim n^{-3/2} 4^n$$

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Comparing with $\mu_{2n} \sim (\Lambda S)^{2n}$, we find that

$$b_\infty \sim \Lambda S$$

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We have argued that at some point b_n becomes constant of order S. What is the solution to the recurrence equation once the b_n's become constant?

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the recurrence equation becomes

$$z\,\phi_n(z) - \delta_{n0} = b(\phi_{n-1}(z) - \phi_{n+1}(z))$$

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and taking the inverse Laplace transform, we find

$$\varphi_0(t) = \frac{J_1(2bt)}{bt}$$

where J_1 is the (first) Bessel function of the first kind.

Using the recurrence relation, we find in general

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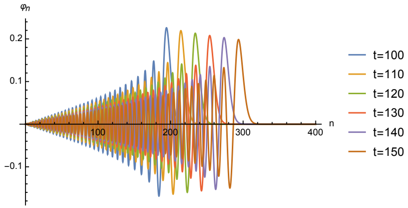

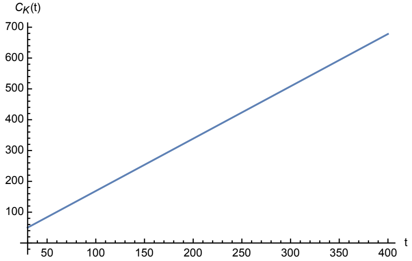

Conclusions: The Big Picture

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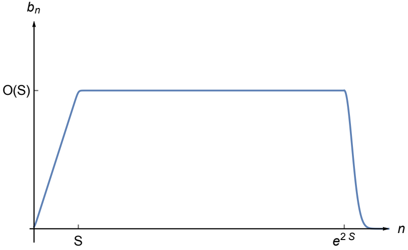

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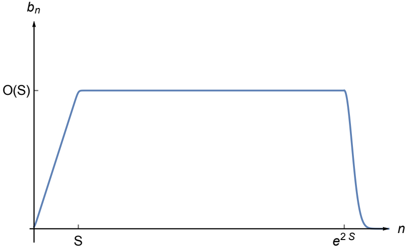

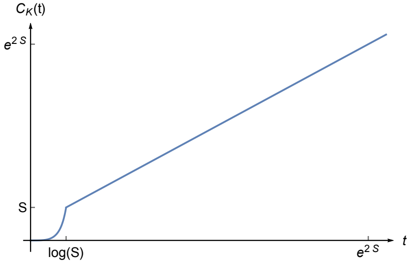

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