Lifting BPS States on K3 and Mathieu Moonshine

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K3 surface is of great interest in physics and mathematics. It has played an important role in compactifications of string theory. The non-linear σ-model on K3 provides a well-studied instance of the AdS_3/CFT_2 correspondence where its symmetric product orbifold is the dual CFT.

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K3 σ-model is also of importance through its role in Mathieu moonshine. [Ooguri '89; Eguchi, Ooguri, Tachikawa '10]

Decomposition of the elliptic genus of K3 ($\mathcal{E}_{\mathrm{K3}}$) into the characters of the small $\mathcal{N}=4$ algebra gives multiplicities which correspond to dimensions of representations of the Mathieu group $\mathbb{M}_{24}$:

[Eguchi, Ooguri, Taormina, Yang '88]

$$\mathcal{E}_{\mathrm{K3}} = 20\mathrm{ch}_{\frac{1}{4},0} - 2\mathrm{ch}_{\frac{1}{4},\frac{1}{2}} + 90\mathrm{ch}_{1,0} + 462\mathrm{ch}_{2,0} + 1540\mathrm{ch}_{3,0} + \cdots \ ,$$

where

$$\mathbf{90} = \mathbf{45} \oplus \overline{\mathbf{45}} \ , \quad \mathbf{462} = \mathbf{231} \oplus \overline{\mathbf{231}} \ , \quad \mathbf{1540} = \mathbf{770} \oplus \overline{\mathbf{770}} \ , \cdots \ .$$

Matthieu moonshine

Characters of these representations satisfy expected modular transformation properties.

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Even though the explicit construction of the moonshine module $V^{\mathbb{M}_{24}} = \bigoplus_{n \geq 1} V_n^{\mathbb{M}_{24}}$ is established [Gannon '12], it is not natural one. The goal of understanding $\mathbb{M}_{24}$ moonshine is to find a natural construction, namely a vertex operator algebra or CFT with automorphism group $\mathbb{M}_{24}$ whose elliptic genus reproduces that of K3.

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An obvious approach to such a natural construction is to use the non-linear σ-model of K3: structure of BPS states at generic points on the moduli space.

Outline

- K3 σ-models
- Lifting of $\frac{1}{4}$-BPS states
- Matthieu moonshine

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Only in regions with enhanced symmetries we know explicit constructions, *e.g.* for the case where K3 is a Kummer surface, namely a torus orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

K3 moduli space

Away from these points, we only know the spectrum of states with high enough supersymmetry: chiral ring. The space of $\frac{1}{2}$-BPS states is protected, their number is constant across the moduli space with multiplicities given by the hodge diamond of K3.

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As we move around in the moduli space, two $\frac{1}{4}$-BPS states may pair up and form a non-BPS state and leave the elliptic genus.

Generic points in moduli space

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If we move away from the orbifold point, 12 of those states are expected to get lifted. We confirm this expectation by performing perturbation theory:

$$102 = (90 + 6) + 6$$
$$90 = (90 + 6) - 6$$

Small $\mathcal{N} = (4, 4)$ superconformal algebra

Generators of holomorphic superconformal algebra ($c = 6$): L_n, $G_r^{\mu\nu}$, and J_n^i (similar for anti-holomorphic generators).
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Holomorphic complex bosons and complex fermions on $\mathbb{T}^4$ are:

$$\Psi^i \ , \quad \bar{\Psi}^i \ , \qquad \partial X^i \ , \quad \partial \bar{X}^i \ , \qquad i = 1, 2 \ ,$$

with (anti-)commutation relations

$$[\partial X_m^{(i)}, \partial \bar{X}_n^{(j)}] = m\delta^{ij}\delta_{m,-n} \ , \quad \{\Psi_r^{(i)}, \bar{\Psi}_s^{(j)}\} = \delta^{ij}\delta_{r,-s}$$

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Small $\mathcal{N} = 4$ primaries $|\phi\rangle$ are defined in the NS sector as:

$$L_n|\phi\rangle = 0 \ , \quad G_r|\phi\rangle = 0 \ , \quad J_n|\phi\rangle = 0 \ , \qquad n \in \mathbb{Z}_{>0} \ , r \in \mathbb{Z}_{>0} + \frac{1}{2} \ .$$

Kummer surface: $\mathbb{T}^4/\mathbb{Z}_2$

At the $\mathbb{Z}_2$ orbifold point associated with the Kummer locus the symmetry algebra is larger: there are 6 fermionic bilinear fields where 3 of them are the R-symmetry currents $J^{\pm,3}$ which generate the $SU(2)$ R-symmetry. The other 3 fields, $\hat{J}^{\pm,3}$, generate the $SU(2)$ flavour symmetry. The latter are lifted when perturbing away from the orbifold point. [Eguchi, Ooguri, Taormina, Yang '89]

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We work with the elliptic genus which is defined as the trace over the $\mathrm{NS}\otimes\tilde{\mathrm{R}}$ sector of the K3 σ-model with the insertion of the fermion number operator F_R:

$$\mathcal{E}^{\mathrm{NS}}_{\mathrm{K3}}(\tau,z)=\mathrm{tr}_{\mathrm{NS}\tilde{\mathrm{R}}}\left(q^{L_0-\frac{1}{4}}y^{J_0}\bar{q}^{\tilde{L}_0-\frac{1}{4}}(-1)^{F_R}\right)\,,\qquad q=e^{2\pi i\tau}\,,\ \ y=e^{2\pi i z}\,.$$

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Elliptic genus counts anti-holomorphic Ramond ground states and non-BPS holomorphic states. We study states with conformal dimensions $h_{\mathrm{NS}}=1$ and $\bar{h}_{\tilde{\mathrm{R}}}=\frac{1}{4}$: there are 102 such states.

Kummer surface: $\mathbb{T}^4/\mathbb{Z}_2$

Small $\mathcal{N} = 4$ algebra has two short (BPS) representations with R-symmetry spin $l = 0$ (singlet) and $l = \frac{1}{2}$ (doublet), and one family of long representations.

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Denote space of $\mathcal{E}_{\mathrm{K3}}^{\mathrm{NS}}$ by V_n^{BPS} with $(h_{NS}, \tilde{h}_R) = (n, \frac{1}{4})$, $n \geq 1$. This splits into the untwisted sector, U_n, and the twisted sector, T_n.

V_n^{BPS} has two possible anti-holomorphic short representation:

$$V_n^{\mathrm{BPS}} = U_n^{\tilde{l}=0} \oplus U_n^{\tilde{l}=\frac{1}{2}} \oplus T_n^{\tilde{l}=0} \oplus T_n^{\tilde{l}=\frac{1}{2}}$$

where n is the holomorphic dimension and $\tilde{l}$ is the anti-holomorphic short representation.

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where n is the holomorphic dimension and $\tilde{l}$ is the anti-holomorphic short representation.

We analyze the states in $V_1^{\mathrm{BPS}}=U_1^{\tilde{l}=0}\oplus U_1^{\tilde{l}=\frac{1}{2}}\oplus T_1^{\tilde{l}=0}\oplus T_1^{\tilde{l}=\frac{1}{2}}$.

Untwisted sector spectrum

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Spectrum of orbifold invariant states in the untwisted sector states with $h_{NS} = 1$ is: $U_1^{\tilde{l}=0} = 0$.
6 $\mathcal{N} = 4$ primary states contribute to $U_1^{\tilde{l}=\frac{1}{2}}$: 3 holomorphic

$$\begin{aligned}
&\Psi^1_{-1/2}\bar{\Psi}^2_{-1/2}|0\rangle_{\mathrm{NS}} \otimes |0\rangle_{\widetilde{\mathrm{NS}}} \ , \\
&\bar{\Psi}^1_{-1/2}\Psi^2_{-1/2}|0\rangle_{\mathrm{NS}} \otimes |0\rangle_{\widetilde{\mathrm{NS}}} \ , \\
&(-\Psi^1_{-1/2}\bar{\Psi}^1_{-1/2} + -\Psi^2_{-1/2}\bar{\Psi}^2_{-1/2})|0\rangle_{\mathrm{NS}} \otimes |0\rangle_{\widetilde{\mathrm{NS}}} \ ,
\end{aligned}$$

and 3 non-holomorphic states

$$\begin{aligned}
&\Psi^1_{-1/2}\bar{\Psi}^2_{-1/2}|0\rangle_{\mathrm{NS}} \otimes \tilde{\Psi}^{(1)}_{-1/2}\tilde{\Psi}^{(2)}_{-1/2}|0\rangle_{\widetilde{\mathrm{NS}}} \ , \\
&\bar{\Psi}^1_{-1/2}\Psi^2_{-1/2}|0\rangle_{\mathrm{NS}} \otimes \tilde{\Psi}^{(1)}_{-1/2}\tilde{\Psi}^{(2)}_{-1/2}|0\rangle_{\widetilde{\mathrm{NS}}} \ , \\
&(-\Psi^1_{-1/2}\bar{\Psi}^1_{-1/2} + -\Psi^2_{-1/2}\bar{\Psi}^2_{-1/2})|0\rangle_{\mathrm{NS}} \otimes \tilde{\Psi}^{(1)}_{-1/2}\tilde{\Psi}^{(2)}_{-1/2}|0\rangle_{\widetilde{\mathrm{NS}}} \ ,
\end{aligned}$$

which correspond to holomorphic flavour symmetry currents $\hat{J}^{3,\pm}$.

Twisted sector spectrum

Toroidal orbifold $\mathbb{T}^4/\mathbb{Z}_2$ has 16 fixed points: $\beta \in \mathbb{Z}_2^4$. For a given fixed point sector β, denote the twisted ground state by $|\sigma^{--}\rangle$. For $\mathcal{E}_{\mathrm{K3}}^{\mathrm{NS}}$, left-movers are in twisted NS sector where there are fermionic zero modes: NS sector twisted ground state is degenerate.

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Spectrum of orbifold invariant twisted sector states:

$$
\begin{aligned}
T_1^{\tilde{l}=\frac{1}{2}} &= 0 \ , \\
T_1^{\tilde{l}=0} &= |\bar{1}2\rangle \ , \quad |\bar{2}1\rangle \ , \quad |11\rangle \ , \quad |22\rangle \ , \\
&\qquad \tfrac{1}{\sqrt{2}}(|\bar{1}1\rangle - |\bar{2}2\rangle) \ , \ \ \tfrac{1}{\sqrt{2}}(|21\rangle + |12\rangle) \ ,
\end{aligned}
$$

where

$$
|ij\rangle \equiv \sqrt{2}\partial X^{(i)}_{-1/2}\Psi^{(j)}_0|\sigma^{--}\rangle \ , \qquad |\bar{\imath}j\rangle \equiv \sqrt{2}\partial \bar{X}^{(i)}_{-1/2}\Psi^{(j)}_0|\sigma^{--}\rangle \ .
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Conformal perturbation theory

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First order perturbation vanishes at first order: conformal weight of BPS states saturates the unitarity bound. If we perturb by λ, then $h(0)$ has to be a minimum of $h(\lambda)$. Thus, we need to go to second order in conformal perturbation theory:

$$\frac{\lambda^2}{2} \int d^2 z_2 d^2 z_3 \langle \varphi^\dagger(z_1, \bar{z}_1) \mathcal{O}(z_2, \bar{z}_2) \mathcal{O}(z_3, \bar{z}_3) \varphi(z_4, \bar{z}_4) \rangle \ .$$

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Using the Möbius transformation, we have

$$\pi \lambda^2 \log \left(\frac{|z_1 - z_4|^2}{\epsilon^2} \right) \frac{1}{(z_1 - z_4)^{2h_\varphi} (\bar{z}_1 - \bar{z}_4)^{2\bar{h}_\varphi}} \int d^2 x \langle \varphi^\dagger(\infty) \mathcal{O}(1) \mathcal{O}(x) \varphi(0) \rangle \ .$$

Numerical coefficient of $\log |z_1 - z_4|$, namely the x integral of the 4-point function, gives the shift of conformal dimension.

Regularisation

We regularise the integrals

$$\frac{\lambda^2}{2}\int d^2z_2 d^2z_3 \langle \varphi^\dagger(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\varphi(z_4)\rangle$$

by cutting out ϵ-discs around insertion points.

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This condition is satisfied for states we study in both untwisted and twisted sectors.

Lifting the untwisted sector: holomorphic states

For $\varphi = \hat{J}^{3,\pm}_{-1}|0\rangle_{\rm NS} \otimes |0\rangle_{\widetilde{\rm NS}}$, one method to compute the lifting matrix is using the general formula derived for lifting of higher-spin currents [Gaberdiel, Peng, IZ, '15]:

$$\gamma^{k\ell} = \lambda^2 \pi^2 \sum_{m=1-s}^{s \bmod 1} (-1)^{\lceil s \rceil - 1 - \lfloor m \rfloor} \binom{2s-2}{s-m-1} \langle \mathcal{O} | \varphi^{(s)k}_{-m} \, \varphi^{(s)\ell}_{m} | \mathcal{O} \rangle \ ,$$

where $s = 1$. We find

$$\gamma^{k\ell} = \frac{\lambda^2 \pi^2}{2} \delta^{k\ell} \ .$$

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Another method is to use our regularisation scheme to compute

$$\int d^2x \langle \varphi^k(\infty)\mathcal{O}^\dagger(1)\mathcal{O}(x)\varphi^\ell(0)\rangle = \int d^2x \left(\frac{1}{(1-x)^2} + \frac{1}{2x}\right)\frac{1}{(1-\bar{x})^2}\ .$$

Writing the antiholomorphic part as a total derivative, we apply Stokes' theorem $\int_{\partial U} F dz + G d\bar{z} = \int_U (\partial_z G - \partial_{\bar{z}} F)\, dz \wedge d\bar{z}$

around $x = 0, 1$, and ∞ and find the same result.

Lifting the untwisted sector: non-holomorphic states

For $\varphi = \hat{J}^{3,\pm}_{-1}|0\rangle_{\mathrm{NS}} \otimes \tilde{J}^{+}_{-1}|0\rangle_{\widetilde{\mathrm{NS}}}$, we define the modes

$$\varphi_m := V_m\Big(\Psi^I_{-\frac{1}{2}}\Psi^J_{-\frac{1}{2}}|0\rangle\Big)\ , \qquad \tilde{\varphi}_m := V_m\Big(\tilde{\Psi}^{(1)}_{-\frac{1}{2}}\tilde{\Psi}^{(2)}_{-\frac{1}{2}}|0\rangle\Big)$$

and compute the 4-point function

$$\langle\mathcal{O}|\varphi(1)\varphi(x)|\mathcal{O}\rangle = \sum_{m,n} ||\varphi_m\tilde{\varphi}_n|\mathcal{O}\rangle||^2 x^{-m-1}\bar{x}^{-n-1}\ .$$

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This gives

$$\int d^2x\langle\varphi^k(\infty)\mathcal{O}^\dagger(1)\mathcal{O}(x)\varphi^\ell(0)\rangle = \int d^2x\left(\frac{1}{(1-x)^2}+\frac{1}{2x}\right)\frac{1}{(1-\bar{x})^2}\ .$$

and we find

$$\gamma^{k\ell} = \frac{\lambda^2\pi^2}{2}\delta^{k\ell}\ .$$

Lifting the twisted sector

There are 16 fixed point sectors in the twisted sector. In each sector, there are 8 states of weight $(1, \frac{1}{2})$ whose lifting matrix we need to investigate. In total there are $8 \times 16 = 128$ states $|\varphi^i\rangle$

$$|ij;\alpha\rangle = \sqrt{2}\partial X^{(i)}_{-1/2}\Psi^{(j)}_0|\sigma^{--}_\alpha\rangle\ , \qquad |\bar{\imath}j;\alpha\rangle = \sqrt{2}\partial \bar{X}^{(i)}_{-1/2}\Psi^{(j)}_0|\sigma^{--}_\alpha\rangle\ .$$

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Of these 8 states, 6 are $\mathcal{N} = 4$ primary fields and 2 are $\mathcal{N} = 4$ descendants of the chiral primaries. The lifting matrix is defined as

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D is a 128×128 matrix. Because the anti-holomorphic part has enough supersymmetry, we can use superconformal Ward identities to write it as a total derivative. We then use Stokes' theorem to reduce the area integral to a contour integral around the the insertion points $0, 1, \infty$.

Lifting the twisted sector

We define $D \equiv D^{(1)} \otimes D^{(2)}$, where $D^{(1)}$ is a 16×16 matrix, encoding the information of the 16 fixed point sectors and $D^{(2)}$ is the 8×8 matrix encoding $|ij;\alpha\rangle$ and $|\bar{\imath}j;\alpha\rangle$.

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It follows that $D^{(1)}$ is diagonal: unless the state in the exchange channel is in vacuum sector, there are no log terms.

$D^{(2)}$ simplifies even further: it is block diagonal with two 4×4 blocks $D^{(3)}$, which are in turn again block diagonal

$$D^{(2)} = \begin{pmatrix} D^{(3)} & 0 \\ 0 & D^{(3)} \end{pmatrix} , \qquad D^{(3)} = \frac{\pi}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} ,$$

which has eigenvalues $\{\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0\}$.

Lifting the twisted sector

Diagonalizing $D = D^{(1)} \otimes D^{(2)}$, all the states outside of the fixed point sector α are left invariant. In fixed point sector α, 2 descendant states have eigenvalues 0 and are left invariant. The remaining 6 states have eigenvalue

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They are lifted by the exact same amount as their partners in $U_1^{\tilde{l}=1/2}$ and $U_1^{\tilde{l}=0}$, forming a long representation of the small $\mathcal{N} = 4$ algebra.

Mathieu moonshine

The challenge in Mathieu moonshine is to find the action of $\mathbb{M}_{24}$ on the states. To find the action on 90, we need to pick the modulus to be invariant under permutations of the fixed point sectors

[Taormina, Wendland '13]

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Computations for the untwisted states does not change. For the twisted states, the 8×8 matrix $D^{(2)}$ does not change either. The 16×16 matrix $D^{(1)}$, however, will be a linear combination of previous computations: $\langle\varphi_\alpha|\mathcal{O}_\beta(1)\mathcal{O}_\gamma(x)|\varphi_\delta\rangle$. We find

$$D^{(1)} = \frac{1}{16}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} ,$$

which has one eigenvalue 1, corresponding to eigenvector O^s, and 15 eigenvalues 0.

Mathieu moonshine

This is exactly what we expect: namely, the states corresponding to the linear combination $\mathcal{O}^s$ get lifted, whereas the 15 directions orthogonal to it do not. [Gaberdiel, Keller, Paul '16].

It would be interesting to analyse higher $\frac{1}{4}$-BPS states and examine whether they agree with the pattern proposed recently by Taormina and Wendland (2019).

Second order perturbation theory

Naively, take the x integral to be independent of z_2. Then evaluate the z_2 integral. This integral is divergent, regularising it thorugh cutting ϵ-discs around z_1 and z_4:

$$\pi\lambda^2 \log\left(\frac{|z_1 - z_4|^2}{\epsilon^2}\right) \frac{1}{(z_1 - z_4)^{2h_\varphi}(\bar{z}_1 - \bar{z}_4)^{2\bar{h}_\varphi}} \int d^2x \langle \varphi^\dagger(\infty)\mathcal{O}(1)\mathcal{O}(x)\varphi(0)\rangle \ .$$

Numerical coefficient of $\log|z_1 - z_4|$, namely the x integral of the 4-point function, gives the shift of conformal dimension. The x integral is itself divergent and needs to be regularised. One would expect that a change of regularisation scheme may change the constant part of the integral, which would then imply that the shift in conformal dimension is scheme-dependent.

Regularisation

This, however, turns out not to be the case: our original regularisation scheme introduces a z_2 dependence for the x integral. More precisely, we already need to regularise the integrals

$$\frac{\lambda^2}{2}\int d^2z_2 d^2z_3 \langle \varphi^\dagger(z_1)\Phi(z_2)\Phi(z_3)\varphi(z_4)\rangle \ .$$

We regularize the z_3 integral by cutting out ϵ-discs around z_1, z_2, z_4. For instance, for z_4, Möbius transformation, cuts out a disc in the x integral around $x = 0$. The cross-ratio is $x = \frac{(z_3-z_4)(z_2-z_1)}{(z_3-z_1)(z_2-z_4)}$ and so that the x integral depends on z_2 if there are divergences.

Let us assume that the OPE of Φ and φ contains a relevant field ϕ:

$$\mathcal{O}(x)\varphi(0) \sim \frac{1}{x^{1+h_\varphi-h}\bar{x}^{1+\bar{h}_\varphi-\bar{h}}}\phi(0) \ .$$

- The integral around 0 vanishes unless φ and ϕ have the same spin,
- As long as $\Delta \neq \Delta_\varphi$, regularisation of the x integral does not give a $\log|z_1 - z_4|$ term.

Regularisation

This puts the constraint: $h \neq h_\varphi$ or $\bar{h} \neq \bar{h}_\varphi$.

This condition is satisfied for states we study:

• For holomorphic states in untwisted sector $\varphi = \hat{J}^{3,\pm}_{-1}|0\rangle_{\mathrm{NS}} \otimes |0\rangle_{\widetilde{\mathrm{NS}}}$ there is no such OPE simply because $(h_\varphi, \bar{h}_\varphi) = (1, 0)$ but ϕ will be in the twisted sector and hence, non-holomorphic: $\bar{h} \neq \bar{h}_\varphi$.

• For non-holomorphic states $\varphi = \hat{J}^{3,\pm}_{-1}|0\rangle_{\mathrm{NS}} \otimes \tilde{J}^{+}_{-1}|0\rangle_{\widetilde{\mathrm{NS}}}$ is marginal $(h_\varphi, \bar{h}_\varphi) = (1, 1)$. The condition requires that there should be no marginal fields in the OPE, *i.e.* the usual condition at first order.

• For the twisted sector states $|ij\rangle$ and $|\bar{\imath}j\rangle$ with $(h_\varphi, \bar{h}_\varphi) = (1, \frac{1}{2})$, the condition is satisfied.

Similar argument leads to same conclusion for divergences at $x = 1$ and $x = \infty$.