PLANCK SCALE ARITHMETIC GEOMETRY

OF AdS2 BLACK HOLE HORIZONS

EMMANUEL FLORATOS

PHYSICS DEPARTMENT UOA,

INP DEMOKRITOS

TENTH CRETE REGIONAL MEETING

IN STRING THEORY

KOLYMBARI GREECE

15-22 SEPTEMBER 2019

RECENT WORK

- **Modular discretization of the AdS2/CFT1 Holography**

  M.Axenides,E.Floratos,S.Nicolis

  JHEP 1402(2014)109   arXiv:1306.5670

- **Chaotic Information Processing by Extremal Black Holes**

  M.Axenides,E.G.Floratos,S.Nicolis

  Int. J. Mod. Phys. D24 (2015) 1542.0122 arXiv:1504.00483

- **Quantum cat map dynamics on AdS2**

  Minos Axenides, Emmanuel Floratos,
  Stam Nicolis
- Eur.Phys.J.C.(2018)78:412
- arXiv:1608.07845

- **The arithmetic geometry of AdS2 and its continuum limit**

  Minos Axenides,Emmanuel Floratos
  Stam Nicolis
- arXiv:1908.06691

PLAN
OF THE PRESENTATION

1) PLANCK SCALE PHYSICS
2) DISCRETUM vs CONTINUUM

   OR COMPUTABLE vs NON-COMPUTABLE
3) ARITHMETIC UV CUTOFF OF AdS2 BH HORIZONS
4) IR /UV CUTOFF AND SPACE-TIME ORIGIN OF

   FINITE BH ENTROPY FOR AdS2
5) CONTINUUM LIMIT OF THE AdS2

   IR/UV ARITHMETIC GEOMETRY
6) CONCLUSIONS-OPEN ISSUES

PLANCK SCALE PHYSICS

- Lp=10^(-33) cm->Tp=10^(-44)
- Mp=10^19 Gev/c

- Where ?
- BING BANG ->ORIGIN OF TIME
- BH CENTER->SINGULARITY,INFINITE DENSITIES
- RED SHIFT FOR BH OBSERVER AT INFINITY

PROBE OF PLANCK SCALE DISTANCES FROM BH HORIZONS

- PERTURBATIVE QUANTUM GRAVITY BREAKS DOWN
- PERTURBATIVE STRING THEORY BREAKS DOWN
- STRING DUALITIES CONNECT WEAK AND STRONG COUPLING PHASES BH SPECTRA OF PLANCK SCALE MASSES->D-BRANES->M-THEORY UNIFICATION
- STRINGS PRODUCE EXOTIC GEOMETRIES – NON GEOMETRIC BACKGROUNDS
- STRING COSMOLOGY->BOUNCE
- **GRAVITY AND SPACE TIME IS EMERGENT!**
- **NO ANSWER YET FOR PLANCK SCALE PHYSICS**

- AdS/CFT PLANCK SCALE CURVATURES=
- STRONGLY COUPLED STRING THEORY
- ON CFT SIDE FREE FIELD THEORIES BUT THE
- BULK PHENOMENA EXTREMELY COMPLICATED SECTORS OF OPERATORS
- ALSO LOCAL DIFFEOMORPHISM INVARIANCE YET TO BE UNDERSTOOD
- PROGRESS FOR BH INTERIORS CLOSE TO THE HORIZON

CONTINUUM vs DISCRETUM
OR NON-COMPUTABLE vs COMPUTABLE

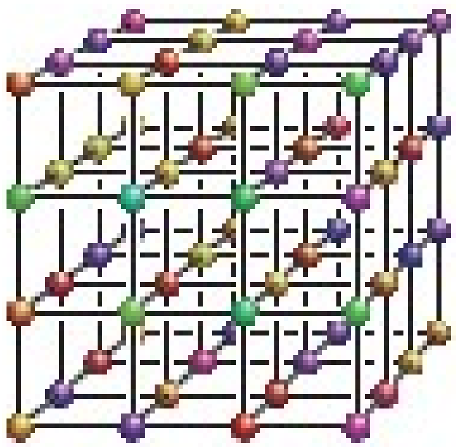
- CONTINUUM LOCAL FIELD THEORIES INFINITE AMOUNT OF INFORMATION PER PLANCK VOLUME
- CONTINUUM STRING THEORY THE SAME!
- BH REMNANT PROBLEM
- EXPERIMENT AT PLANCK SCALE ENERGIES

  PRODUCE MASS Mp BH ‘S ->THEY DO NOT PROBE SPACETIME STRUCTURE AT THE PLANCK SCALE DISTANCE!

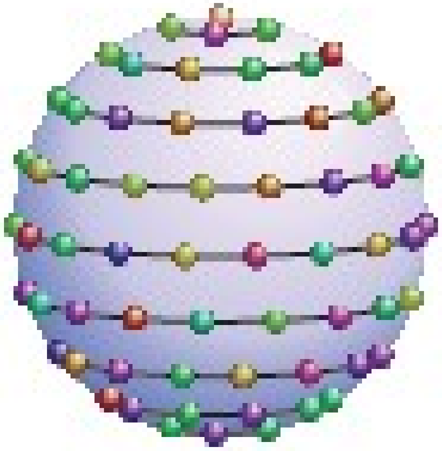
EmQM15
Emergent Quantum Mechanics
OCTOBER 23rd-25th 2015 VIENNA

Physics Utrecht
EMMEΦ

How quantization of gravity leads to a discrete space-time



Gerard 't Hooft

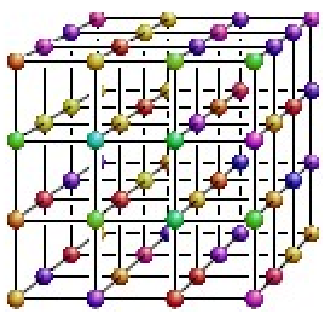


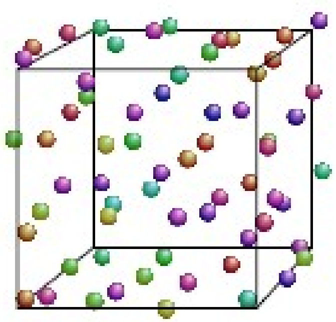
October 25, 2015

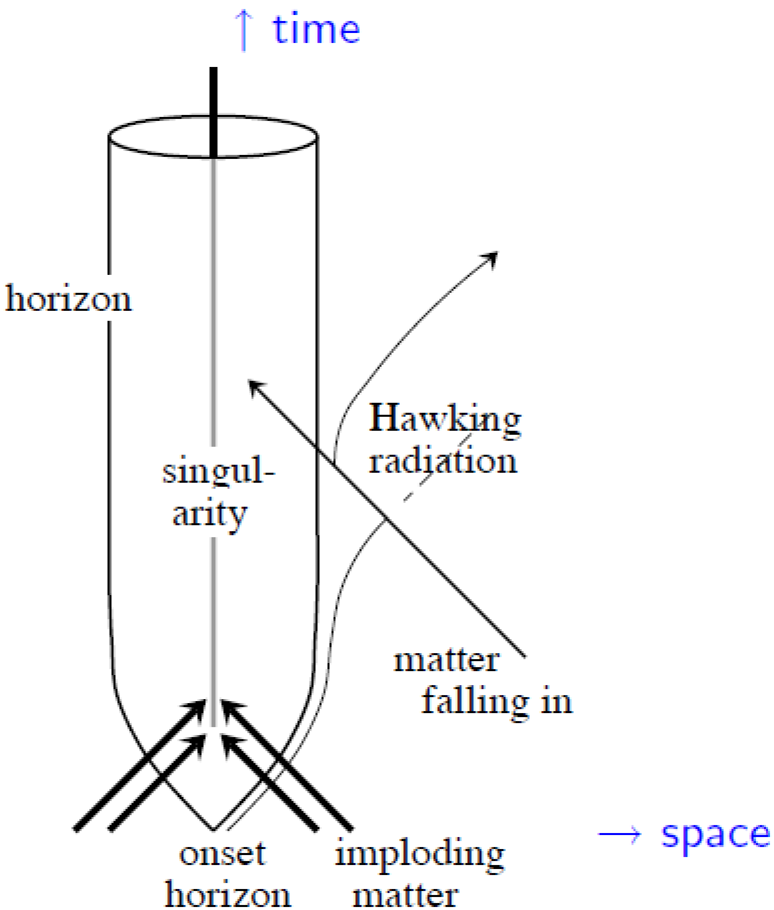
The Planck Units:

$$
\begin{aligned}
h/2\pi \;=\; \hbar &= 1.0546 \times 10^{-34} && \mathrm{kg\ m^2\ sec^{-1}} \\
G_N &= 6.674 \times 10^{-11} && \mathrm{m^3\ kg^{-1}\ sec^{-2}} \\
c &= 2.99792458 \times 10^{-8} && \mathrm{m/sec}
\end{aligned}
$$

$$
\begin{aligned}
L_{\mathrm{Planck}} &= \sqrt{\frac{\hbar G_N}{c^3}} &&= 1.616 \times 10^{-33} && \mathrm{cm} \\
M_{\mathrm{Planck}} &= \sqrt{\frac{\hbar c}{G_N}} &&= 21.76 && \mu\,\mathrm{g} \\
E_{\mathrm{Planck}} &= M_{\mathrm{Planck}} c^2 &&= 1.221 \times 10^{28} && \mathrm{eV}
\end{aligned}
$$







Shapiro shift

Matter going in will gravitationally deform its surrounding space-time.

This effect can be calculated precisely (standard gravity):

Let $p^{\mu}_{\rm in}(\theta,\varphi)$ be the momentum distribution of the in-particles, and

$\delta x^{\mu}_{\rm out}(\theta,\varphi)$ the displacement of the out-partcles, then

$$\tilde\psi^{\pm}_{\rm out}(\kappa) \;=\; e^{-i\kappa\log\frac{8\pi G}{\tilde k^2}}\Big(\tilde A^+(\kappa)\tilde\psi^{\pm}_{\rm in}(-\kappa)+\tilde A^-(\kappa)\tilde\psi^{\mp}_{\rm in}(-\kappa)\Big)$$

$$A^+(\kappa) \;=\; \tfrac{1}{\sqrt\pi}\Gamma(\tfrac12-i\kappa)(\cosh\frac{\pi\kappa}{2}-i\sinh\frac{\pi\kappa}{2})$$

$$A^-(\kappa) \;=\; \tfrac{1}{\sqrt\pi}\Gamma(\tfrac12-i\kappa)(\sinh\frac{\pi\kappa}{2}-i\cosh\frac{\pi\kappa}{2})$$

Thus, in-going waves bounce against a kind of "brick wall"
to become out-going waves. Take inverse Hawking temp. $\beta\to 2\pi$
At given $\tilde k$:

$$e^{-\beta F(\tilde k)} \;=\; \tfrac{1}{\pi\beta}(2\log\Lambda+\log\beta+\gamma+\log(\tilde k^2/8\pi G)))$$

$$U \;=\; \tfrac{\partial}{\partial\beta}(\beta F)\ ,\quad S=\beta(U-F)\ ,\quad \beta\to 2\pi$$

Λ is size of box around black hole.

$\int \mathrm{d}^2\tilde{k} S(\tilde{k})$ diverges ($S(\tilde{k})$ only depends on $\log\log(\tilde{k}^2)$)

To reproduce Hawking's entropy, a cut-off is needed at $|\tilde{k}|$ somewhere near the Planck energy:

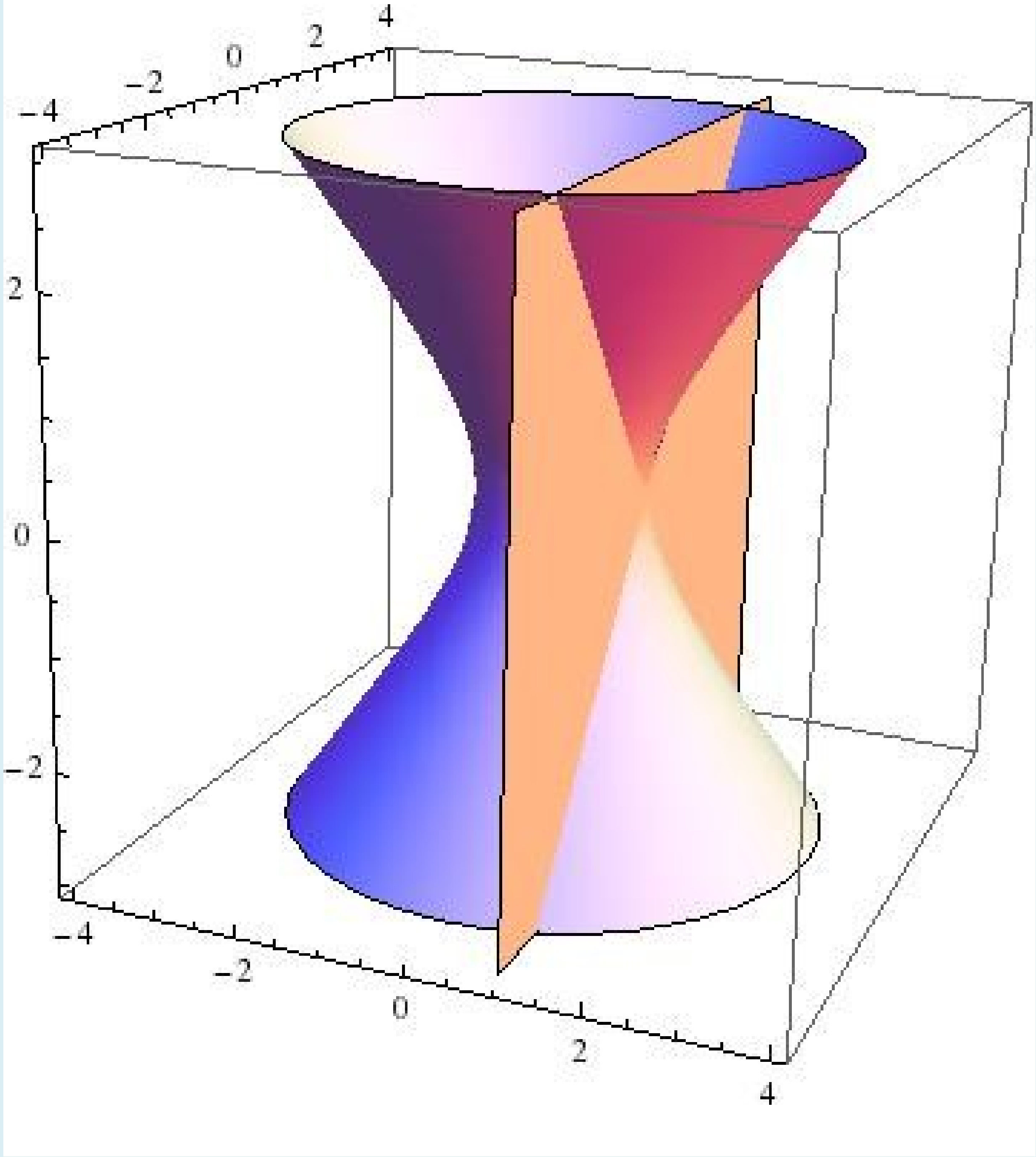
$$|\tilde{k}^2| \le C^2 M_{\mathrm{Planck}}^2 \ , \quad \ell \le C M_{\mathrm{Planck}} r$$

This is a Brillouin zone – a perfect circle due to rotational symmetry. Only a random distribution of points in 2-space can have a circular boundary in $\tilde{k}$ space. *Transverse coordinates* (coord. of bh. horizon) *must be discrete*, and random.

G. Dvali: the *total* number of Hawking particles emitted by black hole: $\approx (M_{BH}/M_{\mathrm{Planck}})^2$, that is, one per unit L_{Planck}^2 of the black hole horizon. No lattice finer that the Planck scale will ever be necessary!

ARITHMETIC UV CUTOFF OF AdS2 BH HORIZONS

- EMBEDDING 2 TIME -1 SPACE MINKOWSKI
- x0^2+x1^2-x2^2=Rads^2=1
- Ruling surface →Light cone rotation around the
- AdS2 throat
- UV CUTOFF
- MINKOWSKI LATTICE   spacing a=Rads/M,
- M integer 1,2,3,4,..
- x0=k a,x1=l a,x2=m a , INTEGRAL AdS2 LATTICE
- k^2+l^2-m^2=M^2
- BREAKING OF SO[2,1,R]→SO[2,1,Z]



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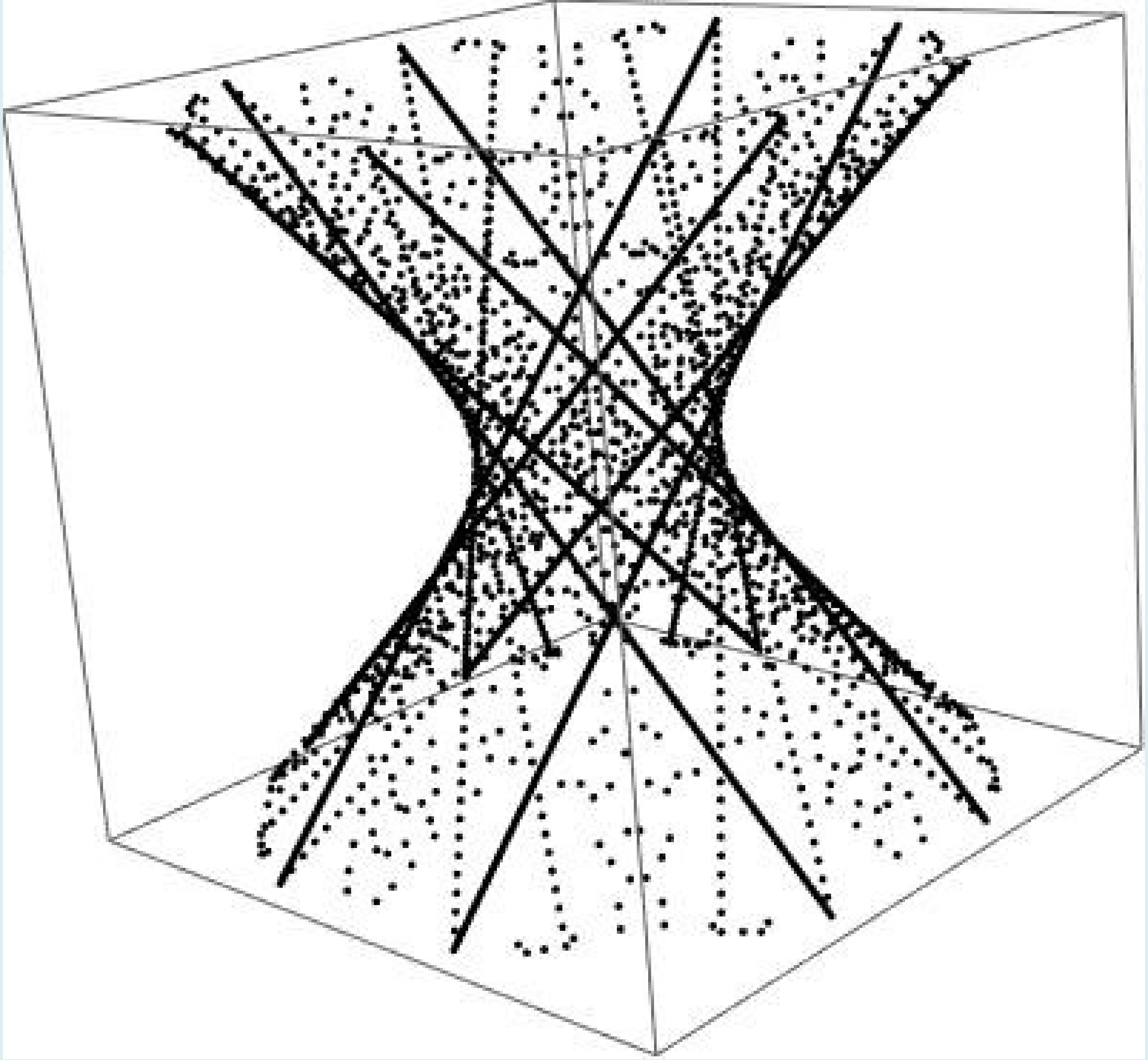


where ϕ is defined by eq. (2.22).

If $\mu = n \in \mathbb{Z}$, we would like to show that $x_0(n)$ and $x_1(n)$ are, also, integers.

This implies that

$$x_0(n) + \mathrm{i}x_1(n) = \frac{k+\mathrm{i}l}{1+\mathrm{i}m}(1+\mathrm{i}n) \tag{2.25}$$

should be a Gaussian integer, i.e. $(1+\mathrm{i}m)/(1+\mathrm{i}n) = a + \mathrm{i}b$ with $a, b \in \mathbb{Z}$.

Therefore

$$1 + \mathrm{i}n = (a - mb) + \mathrm{i}(am + b) \Leftrightarrow \left\{ \begin{array}{l} 1 = a - mb \\ n = am + b \end{array} \right. \tag{2.26}$$

These expressions imply, in turn, that

$$\begin{array}{l} x_0 = k + b(km - l) \\ x_1 = l + b(k + lm) \\ x_2 = n = m + b(1 + m^2) \end{array} \tag{2.27}$$

This completes the dictionary between the rational points on the circle and the integral points on AdS_2. In these expressions b can take any integer value. $\square$

Conversely, any rational point on the unit circle,

$$e^{\mathrm{i}\phi} \equiv \frac{a+\mathrm{i}b}{a-\mathrm{i}b} \Leftrightarrow x_0 + \mathrm{i}x_1 = \frac{a+\mathrm{i}b}{a-\mathrm{i}b}(1+\mathrm{i}n) \tag{2.28}$$

with $a, b \in \mathbb{Z}$, to obtain an integral point, for $\mu = n$, we must have

$$\frac{1+\mathrm{i}n}{a-\mathrm{i}b} = d + \mathrm{i}c \tag{2.29}$$

with $c, d \in \mathbb{Z}$

We immediately deduce that

$$\begin{array}{l} 1 = ad - bc \\ n = ac + bd \end{array} \tag{2.30}$$

These expressions imply that, given the integers a and b, it's possible to find the integers c and d and to express the coordinates x_0, x_1 and x_2 as

$$\begin{array}{l} x_0 = ad + bc \\ x_1 = ac - bd \\ x_2 = ac + bd \end{array} \tag{2.31}$$

The Diophantine equation $1 = ad - bc$ is solved for c and d, given two coprime integers a and b, by the Euclidian algorithm–which seems to lead to a unique solution, implying that the point (x_0, x_1, x_2) is unique.

Therefore there is a one–parameter family of points, labeled by the integer κ:

$$\begin{aligned} x_0 &= ad + bc + 2\kappa ab \\ x_1 &= ac - bd + \kappa(a^2 - b^2) \\ x_2 &= ac + bd + \kappa(a^2 + b^2) \end{aligned} \tag{2.32}$$

We remark, however, that the vector $(2ab, a^2 - b^2, a^2 + b^2)$ is light–like, with respect to the $(+ + -)$ metric: $(2ab)^2 + (a^2 - b^2)^2 - (a^2 + b^2)^2 = 0$. So eq. (2.32) describes a shift of the, original, point $(ad + bc, ac - bd, ac + bd)$, along a light–like direction; and since the shift is linear in the "affine parameter", κ, this generates a light–like line, passing through the original point.

In this way we have established the dictionary between the rational points of the circle and the integral points of the hyperboloid.

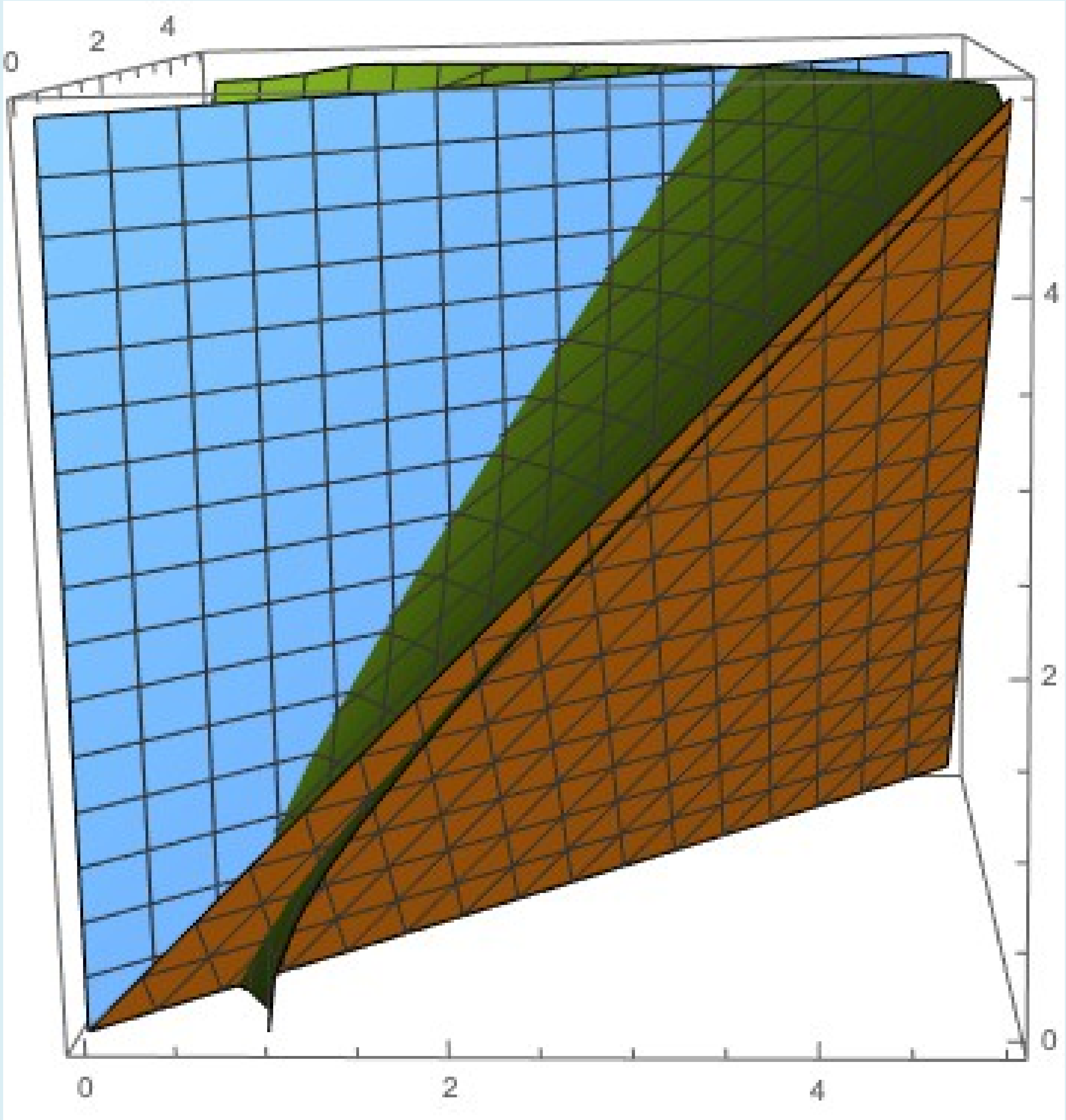
Now we proceed with the study of the discrete symmetries of the integral, $\mathscr{M}^{2,1}$, Lorentzian lattice, where the lattice of integral points on AdS_2 is embedded. $\mathscr{M}^{2,1}$, with one space-like and two time-like dimensions, carries as isometry group the group of integral Lorentz boosts $\mathrm{SO}(2,1,\mathbb{Z})$, as well as integral Poincaré translations. The double cover of this infinite and discrete group is $\mathrm{SL}(2,\mathbb{Z})$, the modular group. This has been shown by Schild [50, 51] in the 1940s. The group $\mathrm{SO}(2,1,\mathbb{Z})$ can be generated by reflections, as has been shown by Coxeter [64], followed by Vinberg [65] and, finally, by Kac in his famous book [52], where he introduced the notion of hyperbolic, infinite dimensional, Lie algebras. The characteristic property of such algebras is that the discrete Weyl group of their root space is an integral Lorentz group. Generalization from $SL(2,\mathbb{Z})$ to other normed algebras has been studied in [66].

The fundamental domain of $\mathrm{SO}(2,1,\mathbb{Z})$ is the minimum set of points of the integral lattice of $\mathscr{M}^{2,1}$,which are not related by any element of the group and from which, all the other points of the lattice can be generated by repeated action of the elements of the group. It turns out that the fundamental region is an infinite set of points and can be generated by repeated action of reflections in the following way:

Using the metric $h \equiv \mathrm{diag}(1,1,-1)$ on $\mathscr{M}^{2,1}$ the generating reflections, elements of $\mathrm{SO}(2,1,\mathbb{Z})$, are given by the matrices

$$\begin{gathered} R_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \\ R_4 = \begin{pmatrix} 1 & -2 & -2 \\ 2 & -1 & -2 \\ -2 & 2 & 3 \end{pmatrix} \end{gathered} \tag{2.33}$$

If (k,l,m) are the coordinates of the integral lattice, the fundamental domain of $\mathrm{SO}(2,1,\mathbb{Z})$ can



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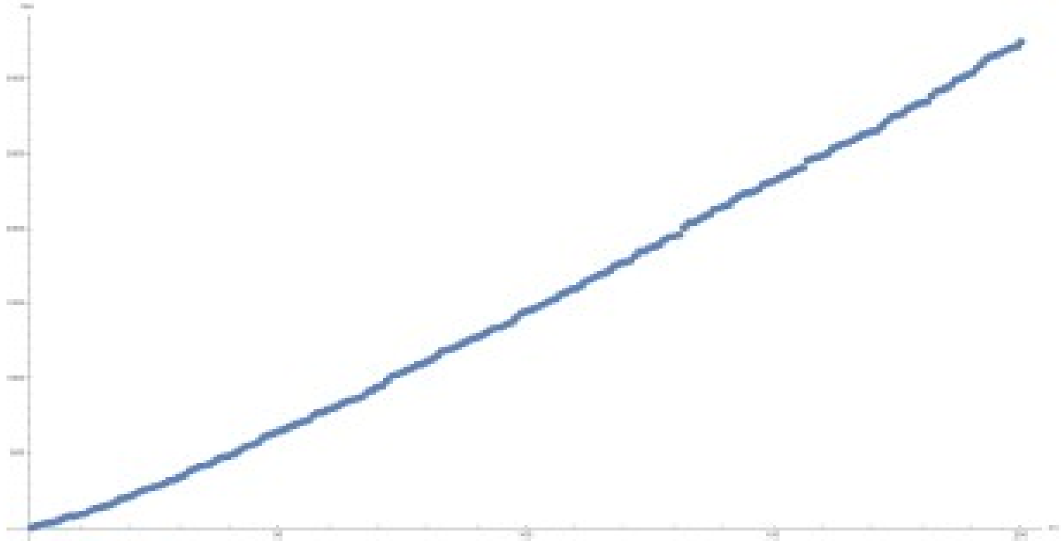


Figure 3: The number of integral points, on AdS_2, as a function of the height, m. Due to symmetry, $m \leftrightarrow -m$, we plot only the positive values of m.

we may repackage these as follows

$$x_0 + \mathrm{i}x_1 = k + \mathrm{i}l = e^{\mathrm{i}\phi}(1 + \mathrm{i}\mu) = e^{\mathrm{i}\phi}(1 + \mathrm{i}m) \Leftrightarrow e^{\mathrm{i}\phi} = \frac{k + \mathrm{i}l}{1 + \mathrm{i}m} \tag{2.21}$$

hence

$$\cos\phi = \frac{k + lm}{1 + m^2} \quad \text{and} \quad \sin\phi = \frac{l - mk}{1 + m^2} \tag{2.22}$$

We remark that these are rational numbers–therefore they label rational points on the circle [63].

The light cone lines at (k, l, m) are, therefore, parametrized as

$$\begin{array}{l} x_0 = \frac{k+lm}{1+m^2} - \mu\frac{l-mk}{1+m^2} \\ x_1 = \frac{l-mk}{1+m^2} + \mu\frac{k+lm}{1+m^2} \\ x_2 = \mu \end{array} \tag{2.23}$$

(When $\mu = x_2 = m$, $x_0 = k$ and $x_1 = l$.)

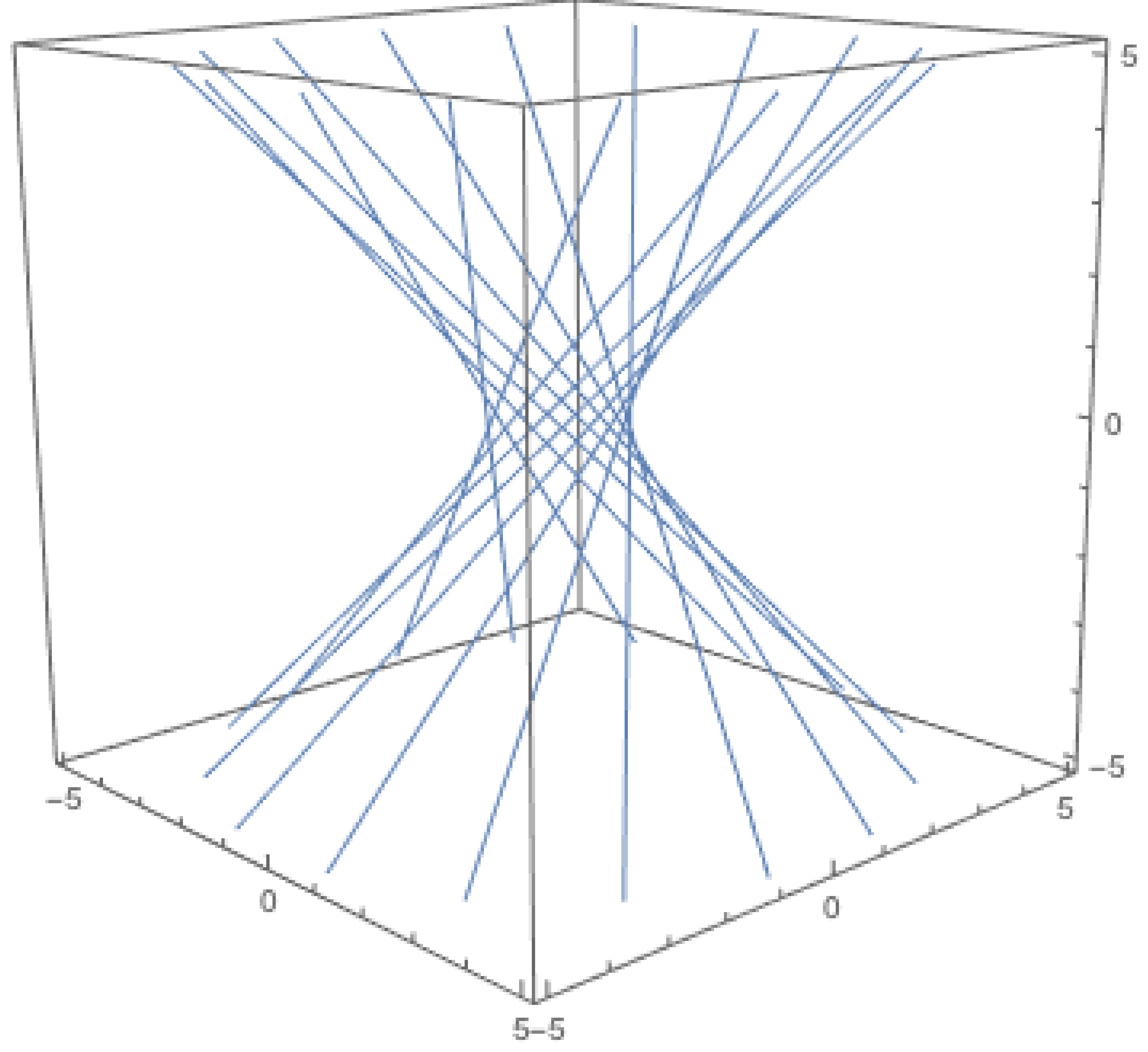
On these specific light-cone lines we shall show that there exist infinitely many integral points, when μ, that labels the space–like direction x_2, takes integer values.

Proof. We write

$$x_0(\mu) + \mathrm{i}x_1(\mu) = e^{\mathrm{i}\phi}(1 + \mathrm{i}\mu) \tag{2.24}$$

IR /UV CUTOFF AND SPACE-TIME ORIGIN OF FINITE BH ENTROPY FOR AdS2

- IR CUTOFF= <u>PERIODIC</u> BOX OF SIZE L
- L=N a, N>M, 1,2,3,4,...
- INTEGRAL AdS2 LATTICE → FOLDED INFINITE MANY TIMES INSIDE THE BOX→ (k,l,m)→(k,l,m)modN
- k^2+l^2-m^2=M^2 modN, k,l,m=0,1,2,...N-1
- FINITE ARITHMETIC GEOMETRY AdS2[Z]->AdS2[Z/NZ]
- SO[2,1,Z]→ SO[2,1,Z/NZ]
- FINITE DIM HILBERT SPACE FOR 1 DOF PER SPATIAL POINT
- dimH=N →FINITE ENTROPY S=N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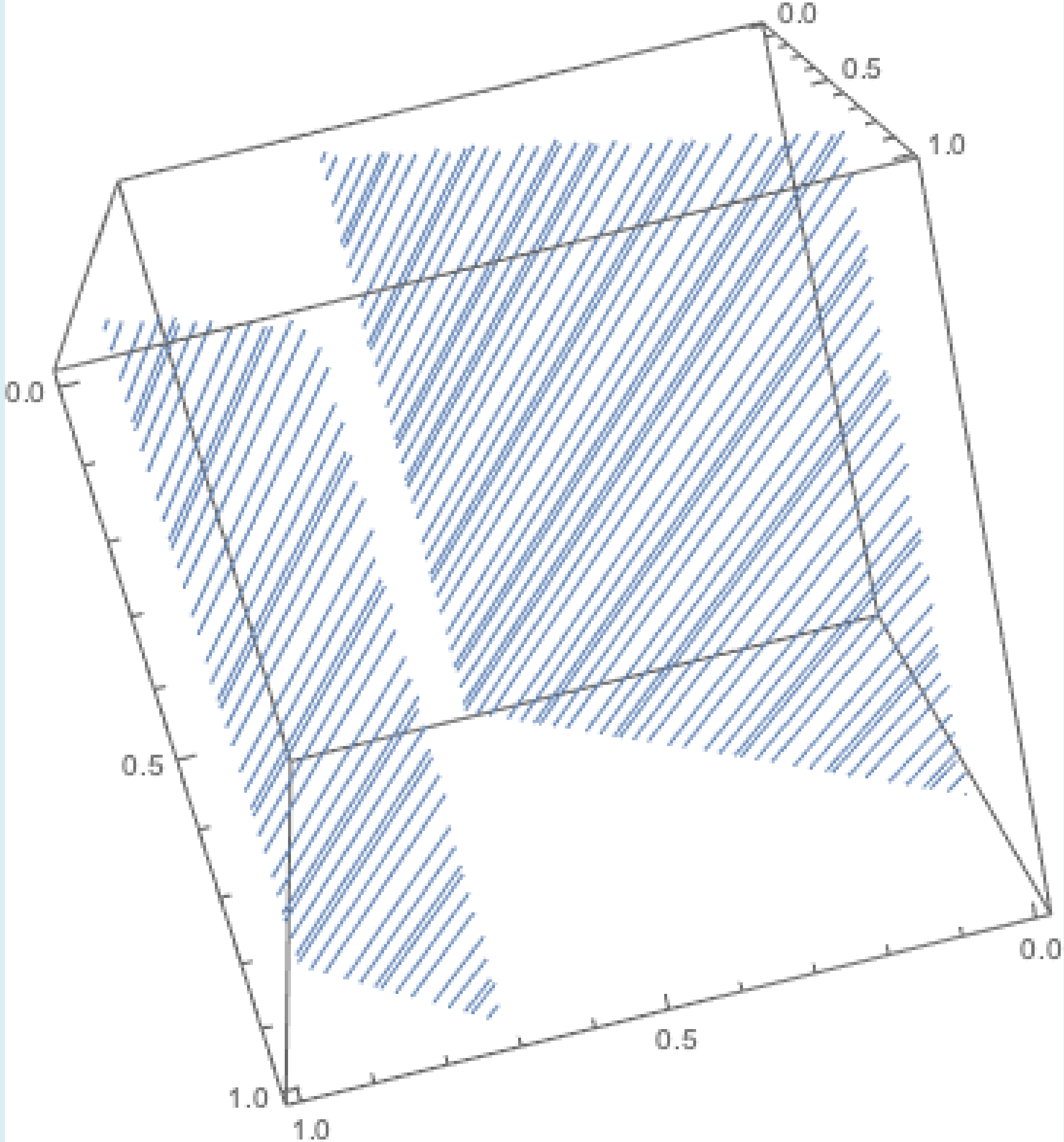
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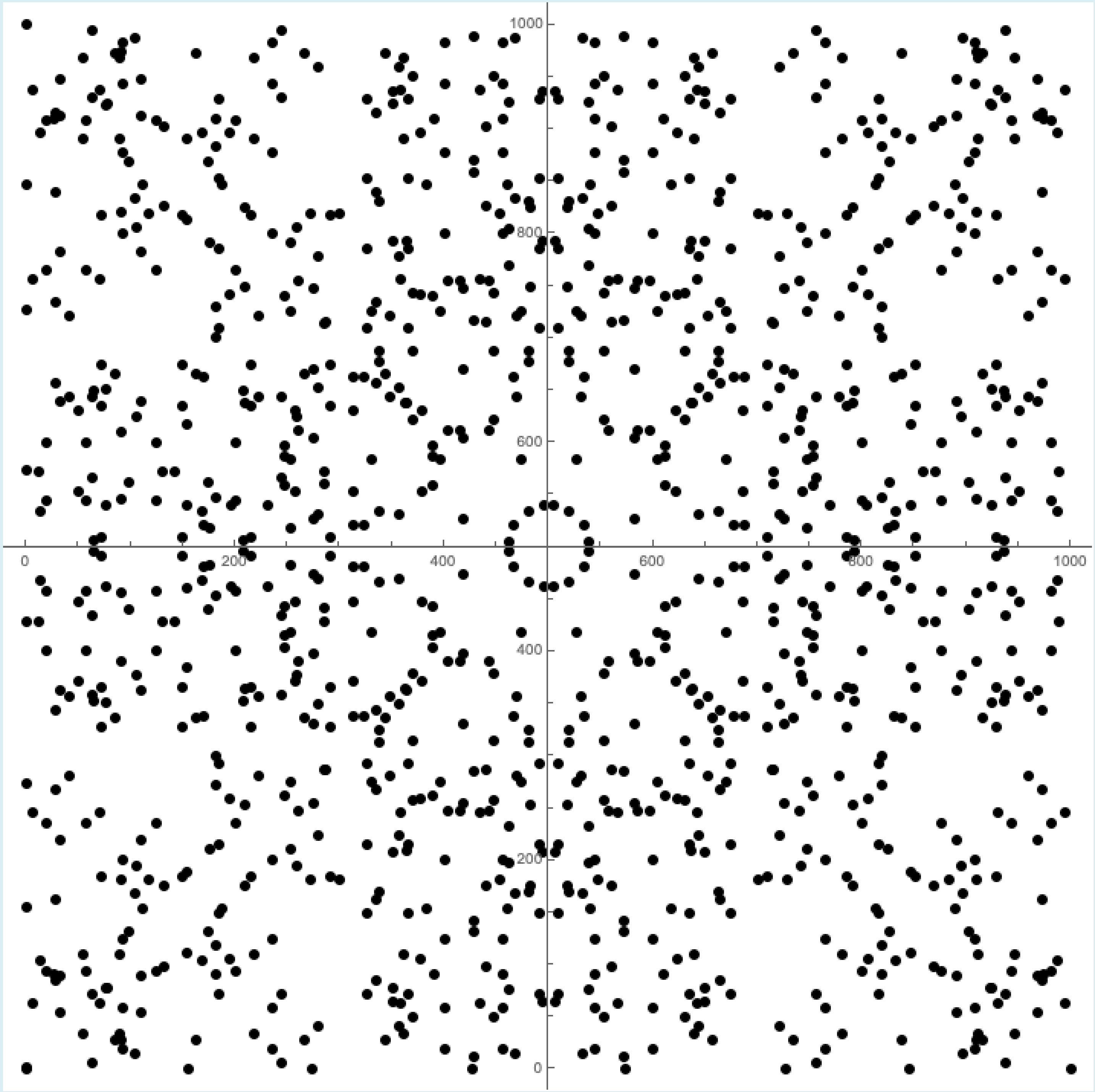

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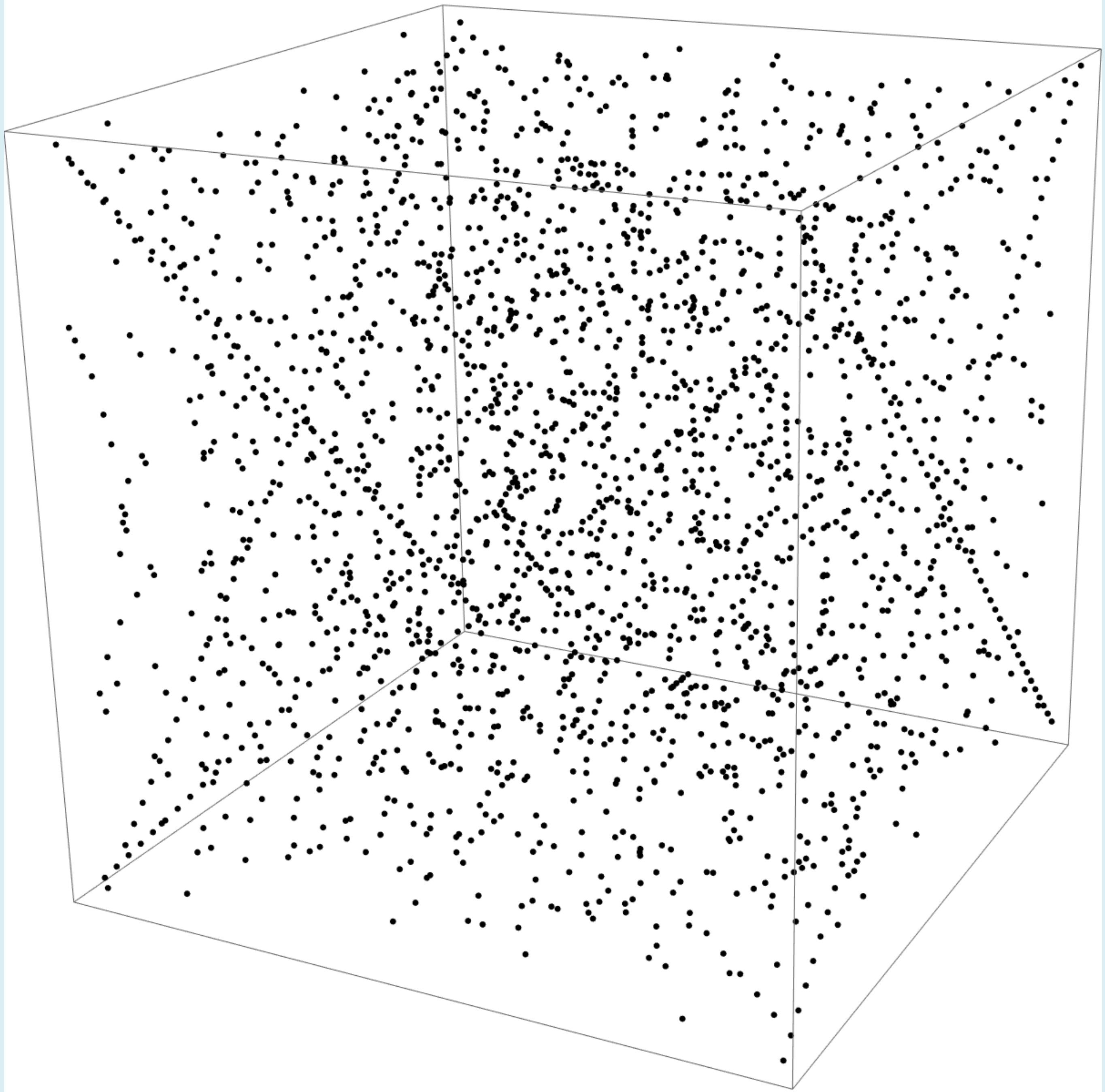


by the ruling parametrization of $\mathrm{AdS}_2[N]$.

3.2 Counting points of $\mathrm{AdS}_2[N]$

The finite geometry, $\mathrm{AdS}_2[p]$, has as isometry group the finite projective modular group, $\mathrm{PSL}_2[p]$. This group is obtained as the reduction mod p, of all elements of $PSL(2,\mathbb{Z})$. The kernel of this homomorphism is the "principal congruent subgroup", Γ_p. The order of $\mathrm{PSL}_2[p]$ is $p(p^2-1)/2$ and the order of its dilatation subgroup is $(p-1)/2$, thus, the number of points of $\mathrm{AdS}_2[p]$ is $p(p+1)$.

It is easy to find the number of points of $\mathrm{AdS}_2[N]$, for any integer N.

Numerical experiments suggest the following recursion relation for the number of points of $\mathrm{AdS}_2[p^k]$, $\mathsf{Sol}(p^k)$,

$$\mathsf{Sol}(p^k) = p^{2(k-1)}\mathsf{Sol}(p) \Rightarrow \mathsf{Sol}(p^k) = p^{2k-1}(p+1) \tag{3.14}$$

where $\mathsf{Sol}(p) = p(p+1)$ and $k = 1, 2, \ldots$ for any prime integer p.

The validity of the above counting can be proved directly by using the coset property of $\mathrm{AdS}_2[p^n]$ and then using factorization of integers for any N [69]. Indeed the rank of the group $\mathrm{PSL}_2[p^n]$ is known to be $p^{2n-2}(p^2-1)/2$ and its dilatation subgroup $\mathrm{PSO}(1,1,p^n)$, $p^{n-1}(p-1)/2$,since it is equal to the number of invertible numbers modulo p^n divided by 2 (due to its projective structure).Thus since $\mathrm{AdS}_2[p^n]$ is identified with the coset geometry $\mathrm{PSL}_2[p^n]/\ \mathrm{PSO}(1,1,p^n)$, we get the promised result, $p^{2n-1}(p+1)$.

For $N = 2^n$ we find $\mathsf{Sol}(2) = 4$, $\mathsf{Sol}(4) = 24$, and $\mathsf{Sol}(2^k) = 4\mathsf{Sol}(2^{k-1})$, for $k \geq 3$. We remark that $N = 4$ is an exception. The solution is $\mathsf{Sol}(2^k) = 2^{2k+1}$, for $k \geq 3$. We display the results of exact enumeration in fig. 9 for $3 \leq N \leq 29$. We notice that there are peaks for composite values of N. Therefore we have many more points inside the box, for $\mathrm{AdS}_2[N]$, than on $\mathrm{AdS}_2[\mathbb{Z}]$. The additional points count the equivalence classes of points of $\mathrm{AdS}_2[\mathbb{Z}]$ mod N.

From these results we deduce that, for large N, the number of solutions, mod N, scales like the area, i.e. N^2. So most of the points of $\mathrm{AdS}_2[N]$ are close to its boundary and holography is possible in this case too [70].

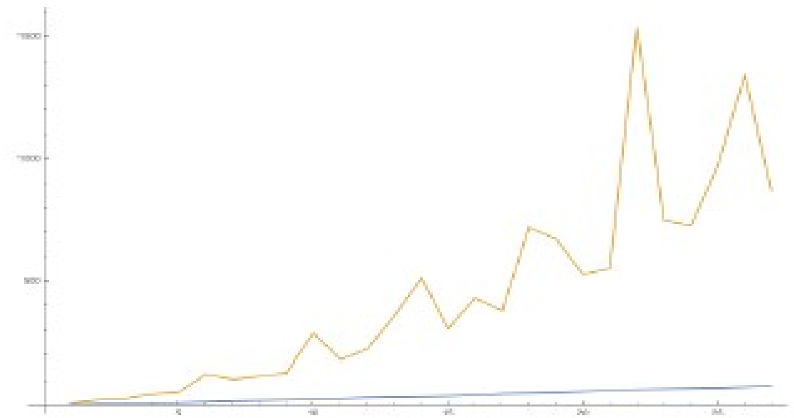


Figure 9: The number of solutions to $k^2+l^2-m^2=1$ (blue curve) and $k^2+l^2-m^2\equiv 1 \bmod N$ (yellow curve), for $3\leq N\leq 29$ obtained by exact enumeration.

4 Continuum limit for large N

4.1 All solutions of $M^2\equiv 1 \bmod N$

In section 2.3, we constructed the discrete geometry $\mathrm{AdS}_2[N]$ by introducing, first, a UV cutoff ($a=R_{\mathrm{AdS}_2}/M$, with M integer) and, also, an IR cutoff $L=Na$, with N another integer, bigger than M.

The continuum limit is defined by any sequence of pairs of integers, (M_n,N_n), $n=1,2,3,\ldots$, such that, for any n, (a) $N_n>M_n$, (b) $M_n^2\equiv 1 \bmod N_n$, and (c) the limit of the ratio N_n/M_n takes a finite value, >1 (as $n\to\infty$), which we can identify with L/R_{AdS_2}.

Below we shall present the general solution to the equation $M^2\equiv 1 \bmod N$. Subsequently, we shall select those solutions that satisfy the other requirements.

The first step is to factor N into (powers of) primes, $N=N_1\times N_2\times\cdots\times N_l=q_1^{k_1}q_2^{k_2}\cdots q_l^{k_l}$. Then the equation $M^2\equiv 1 \bmod N$, is equivalent to the system

$$M_I^2\equiv 1 \bmod q_I^{k_I} \tag{4.1}$$

where $I=1,2,\ldots,l$. The Chinese Remainder Theorem [62] then implies that all the solutions of eq. (4.1) can be used to construct M, with $M=M_1m_1n_1+\cdots M_lm_ln_l$, where $M_I\equiv M \bmod N_I$, $m_I=N/N_I$, $n_I\equiv m_I^{-1} \bmod N_I$.

CONTINUUM LIMIT OF THE AdS2 IR/UV ARITHMETIC GEOMETRY

- Rads=M a   L=N a
- AdS2[Z/NZ]
- ->k^2+l^2-m^2=M^2 mod N, k,l,m=0,1,2,...N-1
- CONTINUUM LIMIT   a→0   COMPLICATED IF IT EXISTS!
- BUT !   CHOOSE,SEQUENCE OF PAIRS OF INTEGERS   M, N SUCH THAT
- 1) N>M
- 2)M^2=1modN
- 3) N/M→ THE WELL DEFINED FINITE RATIO L/Rads=r
- AS   a→0
- INDEED WE FOUND SEVERAL SOLUTIONS (FIBONACCI SEQUENCE + THEIR

4.2 Fibonacci sequences for the UV/IR cutoffs

Although it is easy to demonstrate the existence of such sequences–for example, $N_n = 2^n$ and $M_n = 2^{n-1} \pm 1$, where $M_n^2 \equiv 1 \bmod N_n$ and $N_n/M_n \to 2$, in this section we focus on a particular class of sequences, based on the Fibonacci numbers, f_n [62]. This case is of particular interest, since, in our previous paper [41], where we studied fast scrambling, we found that, for geodesic observers, moving in $\mathrm{AdS}_2[N]$, with evolution operator the Arnol'd cat map, the fast scrambling bound is saturated, when N is a Fibonacci integer.

The Fibonacci sequence, defined by

$$\begin{aligned} &f_0 = 0; f_1 = 1 \\ &f_{n+1} = f_n + f_{n-1} \end{aligned} \tag{4.2}$$

can be written in matrix form

$$\left(\begin{array}{c} f_n \\ f_{n+1} \end{array}\right) = \underbrace{\left(\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right)}_{\mathsf{A}} \left(\begin{array}{c} f_{n-1} \\ f_n \end{array}\right) \tag{4.3}$$

We remark that the famous Arnol'd cat map can be written as

$$\left(\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right) = \mathsf{A}^2 \tag{4.4}$$

Since the matrix A doesn't depend on n, we can solve the recursion relation in closed form, by setting $f_n \equiv C\rho^n$ and find the equation, satisfied by ρ

$$\rho^{n+1} = \rho^n + \rho^{n-1} \Leftrightarrow \rho^2 - \rho - 1 = 0 \Leftrightarrow \rho \equiv \rho_\pm = \frac{1 \pm \sqrt{5}}{2}$$

Therefore, we may express f_n as a linear combination of ρ_+^n and $\rho_-^n = (-)^n \rho_+^{-n}$:

$$f_n = A_+\rho_+^n + A_-\rho_-^n \Leftrightarrow \left\{ \begin{array}{l} f_0 = A_+ + A_- = 0 \\ f_1 = A_+\rho_+ + A_-\rho_- = 1 \end{array} \right. \tag{4.5}$$

whence we find that

$$A_+ = -A_- = \frac{1}{\rho_+ - \rho_-} = \frac{1}{\sqrt{5}}$$

therefore,

$$f_n = \frac{\rho_+^n - (-)^n \rho_+^{-n}}{\sqrt{5}} \tag{4.6}$$

It's quite fascinating that the LHS of this expression is an integer!

The eigenvalue $\rho_+ > 1$ is known as the "golden ratio" (often denoted by ϕ) and it's straightforward to show that $f_{n+1}/f_n \to \rho_+$, as $n \to \infty$.

Furthermore, it can be shown, by induction, that the elements of A^n are, in fact, the Fibonacci numbers themselves, arranged as follows:

$$\mathsf{A}^n = \left(\begin{array}{cc} f_{n-1} & f_n \\ f_n & f_{n+1} \end{array} \right) \tag{4.7}$$

One reason this expression is useful is that it implies that $\det \mathsf{A}^n = (-)^n = f_{n-1}f_{n+1} - f_n^2$.

For $n = 2l+1$, we remark that this relation takes the form $f_{2l+1}^2 = 1 + f_{2l}f_{2l+2}$.

Now, since f_{2l+1} and f_{2l+2} are successive iterates, they're coprime, which implies, that $f_{2l+1}^2 \equiv 1 \bmod f_{2l+2}$.

Therefore, the sequence of pairs, $(M_l = f_{2l+1}, N_l = f_{2l+2})$, where $l = 1, 2, 3, \ldots$, satisfy all of the requirements and the corresponding limiting ratio, L/R_{AdS_2}, can be found analytically. It is, indeed, equal to $\rho_+ = (1+\sqrt{5})/2$, the golden ratio.

We recall here that the periods of the Fibonacci sequence mod N, for any integer N, has been analyzed in the literature (cf. [71]) and, in the case when N, the IR cutoff, is, itself, a Fibonacci integer, then the period of the corresponding Arnol'd cat map grows logarithmically with N and this is the reason for the saturation of the fast scrambling bound [41]. In the next subsection we shall consider the so-called $k-$Fibonacci sequences, which will be important for removing the IR cutoff.

4.3 Generalized k-Fibonacci sequences and UV/IR cutoffs

It's possible to generalize the Fibonacci sequence in the following way:

$$g_{n+1} = kg_n + g_{n-1} \tag{4.8}$$

with $g_0 = 0$ and $g_1 = 1$ and k an integer. This is known as the "$k-$Fibonacci" sequence [72].

We may solve for $g_n \equiv C\rho^n$; the characteristic equation for ρ, now, reads

$$\rho^2 - k\rho - 1 = 0 \Leftrightarrow \rho_\pm(k) = \frac{k \pm \sqrt{k^2+4}}{2} \tag{4.9}$$

and express g_n as a linear combination of the $\rho_\pm$:

$$g_n = A_+\rho_+(k)^n + A_-\rho_-(k)^n = \frac{\rho_+(k)^n - (-)^n\rho_+(k)^{-n}}{\sqrt{k^2+4}} \tag{4.10}$$

that generalizes eq. (4.6).

In matrix form

$$\begin{pmatrix} g_n \\ g_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}}_{\mathsf{A}(k)} \begin{pmatrix} g_n \\ g_{n-1} \end{pmatrix} \tag{4.11}$$

Similarly as for the usual Fibonacci sequence, we may show, by induction, that

$$\mathsf{A}(k)^n = \begin{pmatrix} g_{n-1} & g_n \\ g_n & g_{n+1} \end{pmatrix} \tag{4.12}$$

We find that $\det \mathsf{A}(k)^n = (-)^n$, therefore that $g_{2l+1}^2 \equiv 1 \bmod g_{2l+2}$; thus, $g_{2l+2}/g_{2l+1} \to L/R_{\mathrm{AdS}_2} = \rho_+(k)$, where the eigenvalue of $\mathsf{A}(k)$, $\rho_+(k)$, that's greater than 1, of course, depends on k. At this point we have determined L, the IR cutoff, in terms of R_{AdS_2}. This limiting procedure has removed the UV cutoff, since $a \to 0$, however the IR cutoff, L is, still, present.

What is remarkable is that, using the additional parameter, k, of the $k-$Fibonacci sequence, it is possible to remove the IR cutoff, as well, since it is possible to send $L \to \infty$, as $k \to \infty$, keeping R_{AdS_2} fixed.

If k remains finite, the periodic box cannot be removed and, in the continuum limit, $a \to 0$, we obtain infinitely many foldings of the AdS_2 surface inside the box due to the mod L operation.

The Fibonacci sequence, taken mod N, is periodic, with period $T(N)$; this turns out to be a "random" function of N. The "shortest" periods, as has been shown by Falk and Dyson [71], occur when $N = F_l$, for any l. In that case, $T(F_l) = 2l$.

We may, thus, ask the same question for the $k-$Fibonacci sequence, where the ratio of its successive elements, g_{n+1}/g_n tend to the so-called "$k-$silver ratio",

$$\rho_+(k) = \frac{k + \sqrt{k^2+4}}{2} \tag{4.13}$$

(the "silver ratio" is $\rho_+(k=2)$)

From eq. (4.12), taking mod g_l on both sides, we find that, when $n = l$, the matrix becomes $\pm$(the identity matrix), so $T(g_l) = l$ or $2l$, respectively; thereby generalizing the Falk–Dyson result for the $k-$Fibonacci sequences. Since, for large l, $g_l \sim e^{l \log \rho_+(k)}$, $\log \rho_+(k)$ can be identified with the Lyapunov exponent of the dynamics of the $\mathsf{A}(k)$ map. What is interesting in this generalization is that $\rho_+(k) \sim k$, so, for, large k, can become significantly larger than the Lyapunov exponent of the Arnol'd cat map–so the scrambling time is significantly shorter.

CONCLUSIONS – OPEN ISSUES

- We presented an information-theoretic finite-discretization of the AdS_2 geometry introducing a natural UV and IR cutoff.
- The SL(2,R) isometry is deformed to the finite arithmetic group $SL(2,Z_N)$.
- This group is the isometry group of the $AdS_2[Z_N]$ observers.
- The UV-cutoff is at the sub-Planckian level, while the IR-cutoff is the Planck

- The continuum limit exists for sequences of pairs of UV and IR-cutoffs from the K-Fibonacci integers which are related through the UV/IR correspondence.
- The IR-cutoff can be removed when $K \to \infty$.
- The QM Hilbert space for single particles has a finite dimension equal to the BH entropy and the AdS/CFT correspondence is exact.
- For geodesic observers there is a strong chaotic scrambling of Gaussian wavepackets along the spatial direction of AdS_2.
- The scrambling time bound is saturated for UV/IR cutoffs from generalized K-Fibonacci sequences with ratios (golden/silver).

NEW DIRECTIONS (work in progress)

- The arithmetic modular discretization works with UV/IR cutoffs for $AdS_{3,4,5}$ the BTZ black hole, wormholes and in general for geometries with global algebraic minimal embeddings in flat Minkowski spacetimes.

THANK YOU!

- MOTIVATION ON TOP OF THE ASSUMPTIONS

  1)THE EIGENSTATE THERMALIZATION HYPOTHESIS

  GAUSSIAN PDF OF EIGENSTATE’ S PROB VALUES

  FLAT PDF OF EIGENSTATE’S PHASES

  2) SATURATION OF THE SCRAMBLING TIME BOUND

  3) RELATION OF QUANTUM COMPLEXITY WITH

SIMPLEST EXAMPLE
DIFUSION OF SINGLE PARTICLE WAVE PACKETS ON ADS2
TIME-RADIAL GEOMETRY OF EXTREMAL BH'S

- OLD AND LARGE NEAR EXTREMAL BH'S

- GEOMETRY =AdS2 XΣ , Σ=COMPACT ANGULAR DIRECTIONS

- ADS2 RADIAL MOTION
- AdS2[R]=SL[2,R]/SO[1,1,R]

- DISCRETIZE

- AdS2[N]=SL[2,ZN]/SO[1,1,ZN]

- CONSTRUCT THE AdS2 UNITARY EVOLUTION MATRIX
  OF PROBE STRING BITS
  -> USE SUPERDCONFORMAL QM
  OF SL[2,ZN] ISOMETRY QUANTUM MAPS
  (TOWNSEND-STROMINGER-KALOSH 1998)

EIGENSTATE THERMALIZATION SENARIO
PAGE,DEUTSCH,BERRY,SREDNICKI

IF THE EIGENSTATES OF A CLOSED QM SYSTEM ARE RANDOM

(RANDOM PHASES AND GAUSSIAN DISTRIBUTED AMPLITUDES)

THEN ANY INITIAL PURE STATE OF A SUBSYSTEM THERMALIZES

TO

THE THERMAL DENSITY MATRIX OF THE SUBSYSTEM

RELATION TO THE INFORMATION PARADOX

ARNOLD QUANTUM CAT MAP A={{1,1},{2,1}}

EXACT CONSTRUCTION OF THE SPECTRUM AND EIGENSTATES FOR N=p,prime,

LINEAR SPECTRUM

RANDOM EIGENSTATES ->LINEAR COMBINATIONS OF MULTIPLICATIVE CHARACTERS OF
GF[P]

1)RANDOM PHASES
2)GAUSSIAN RANDOM AMPLITUDES

BUT SCARS(VOROS.,NONEMACHER)F
FOR SEQUENCES OF N's WITH SHORT PERIODS

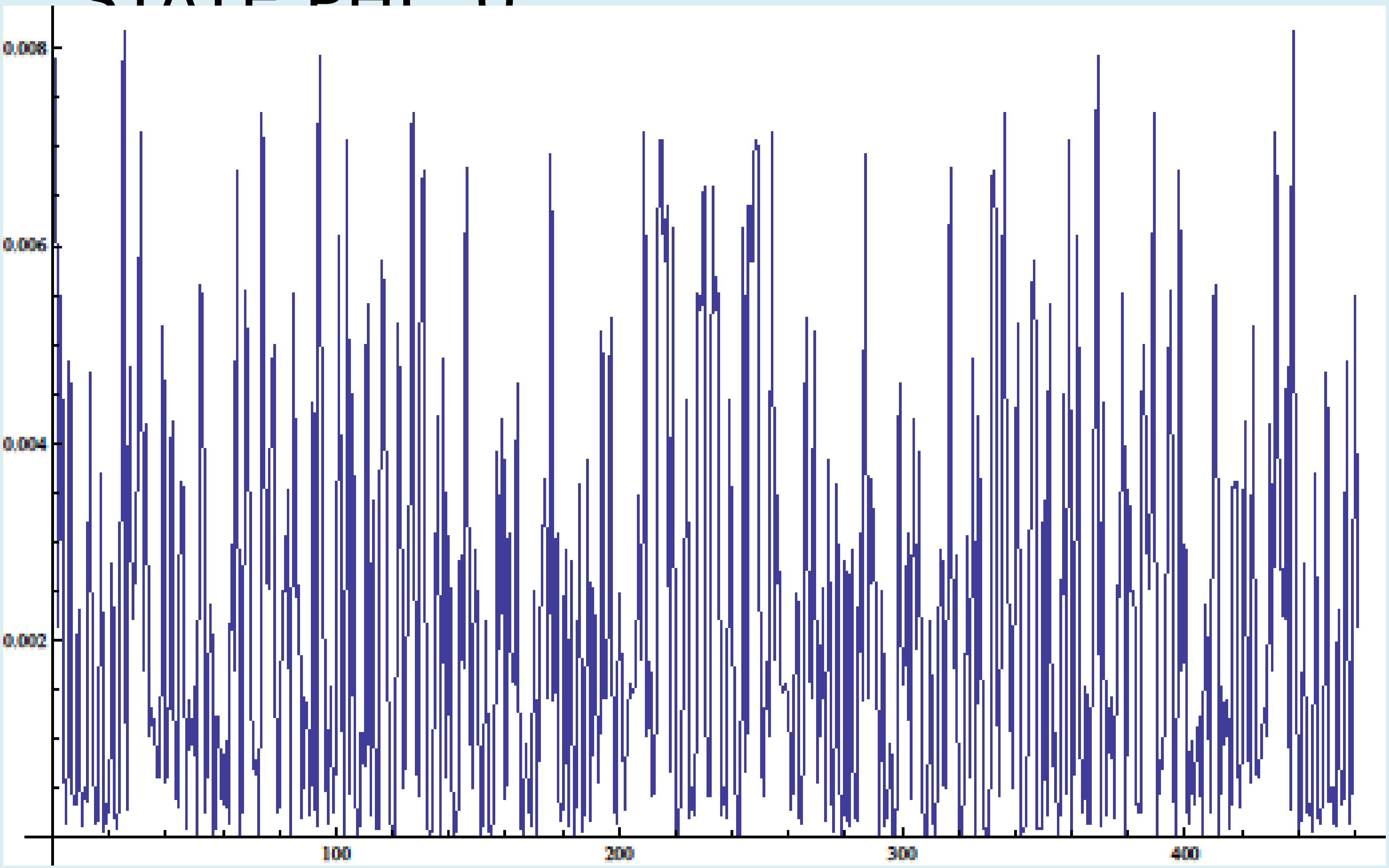
QUANTUM CHAOS, STRONG MIXING

- FACTORIZATION FOR ARNOLD CAT MAPS IMPLIES LOGARITHMIC IMPROVEMENT FROM N^2->N LOGN

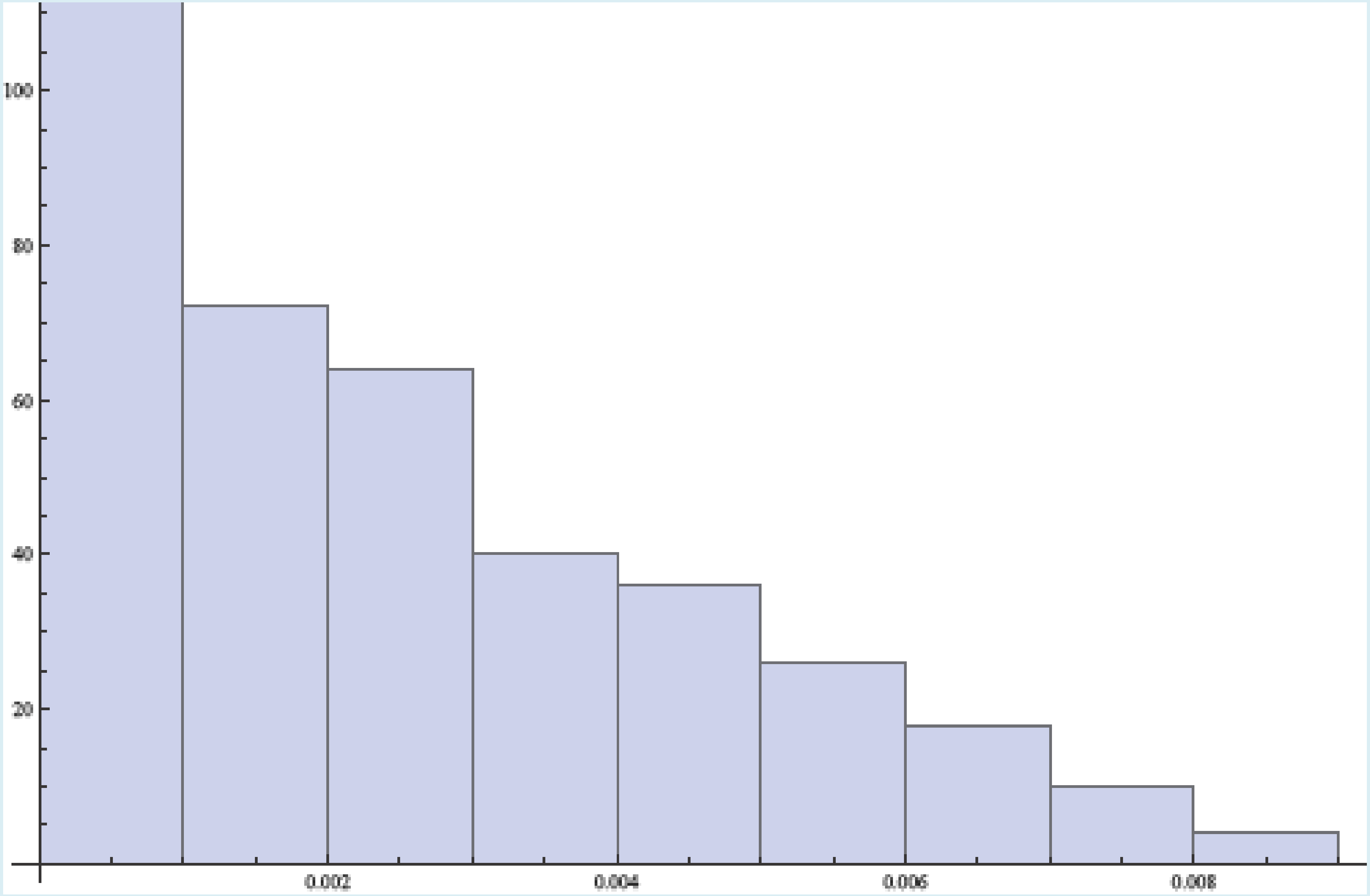
- USING QUANTUM CIRCUITS FOR THE IMPLEMENTATION OF THE QUANTUM MAP AND
COUNTING THE NUMBER OF GATES NLOGN->(LOGN)^2

- EXACTLY AS FOR THE QUANTUM FOURIER FACTORIZATION ALGORITHM OF SHOR.

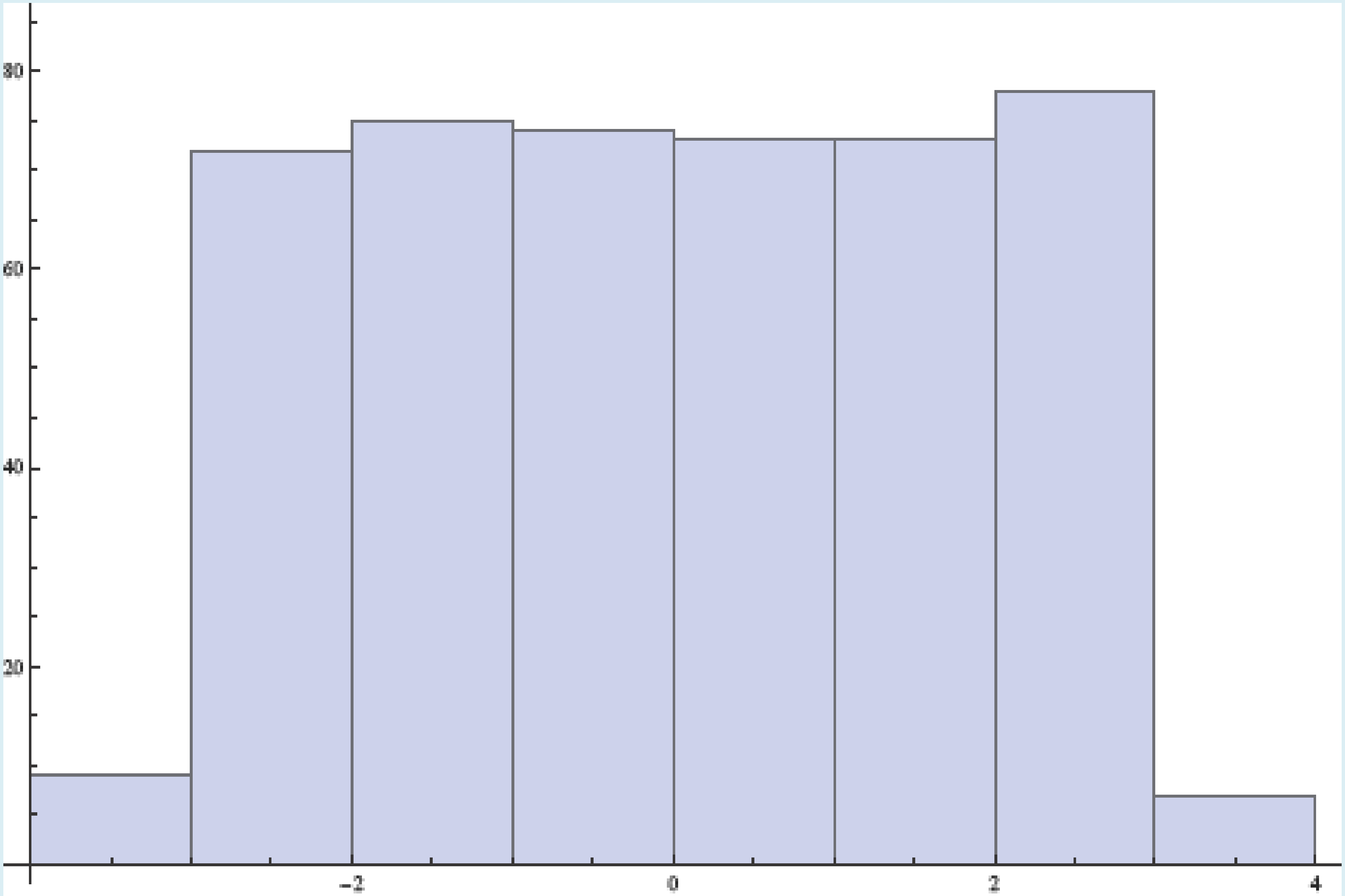
- N=461,T[461]=23 ,GROUND STATE PHI=0



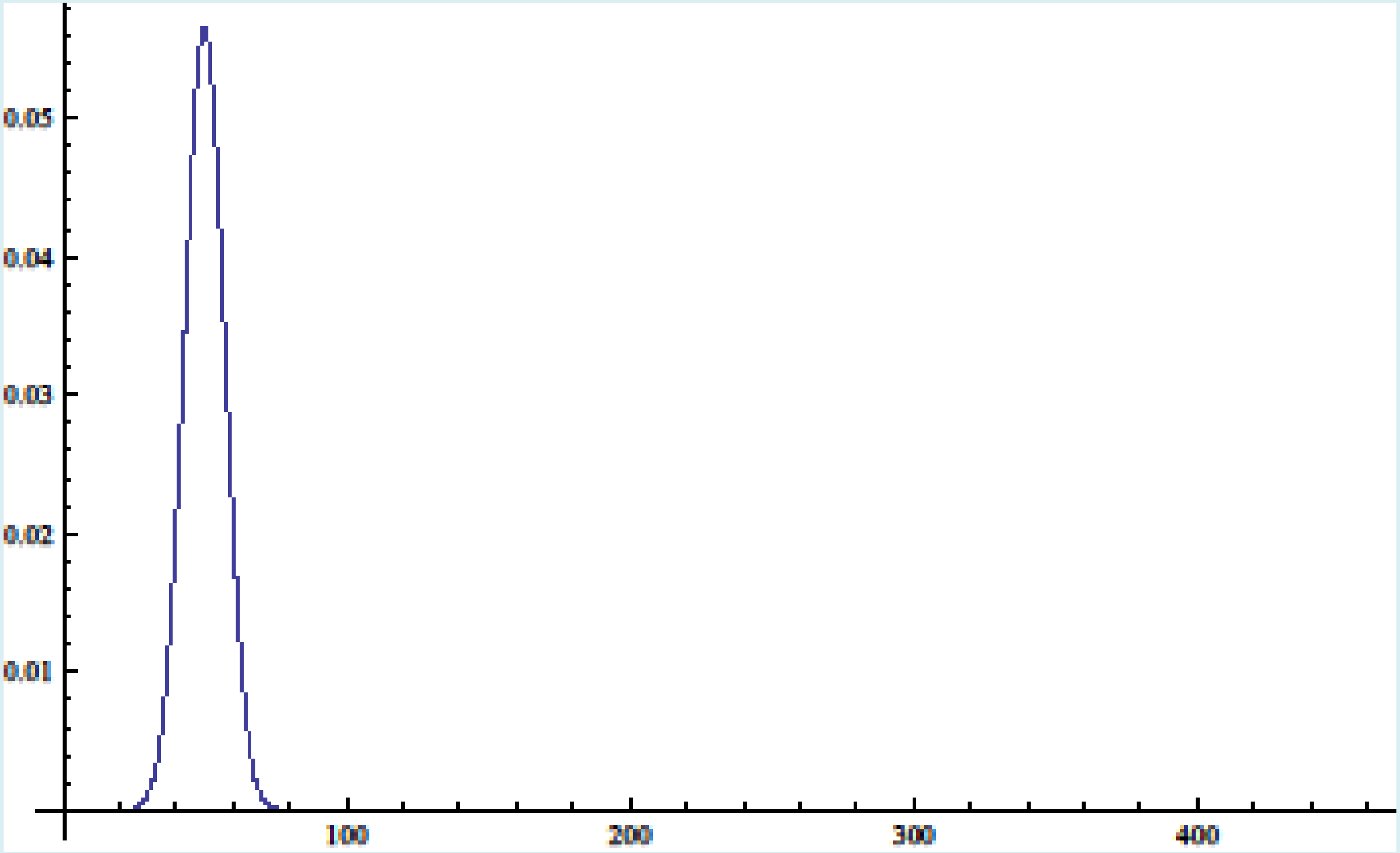
- GROUND STATE AMPLSQUARE DF



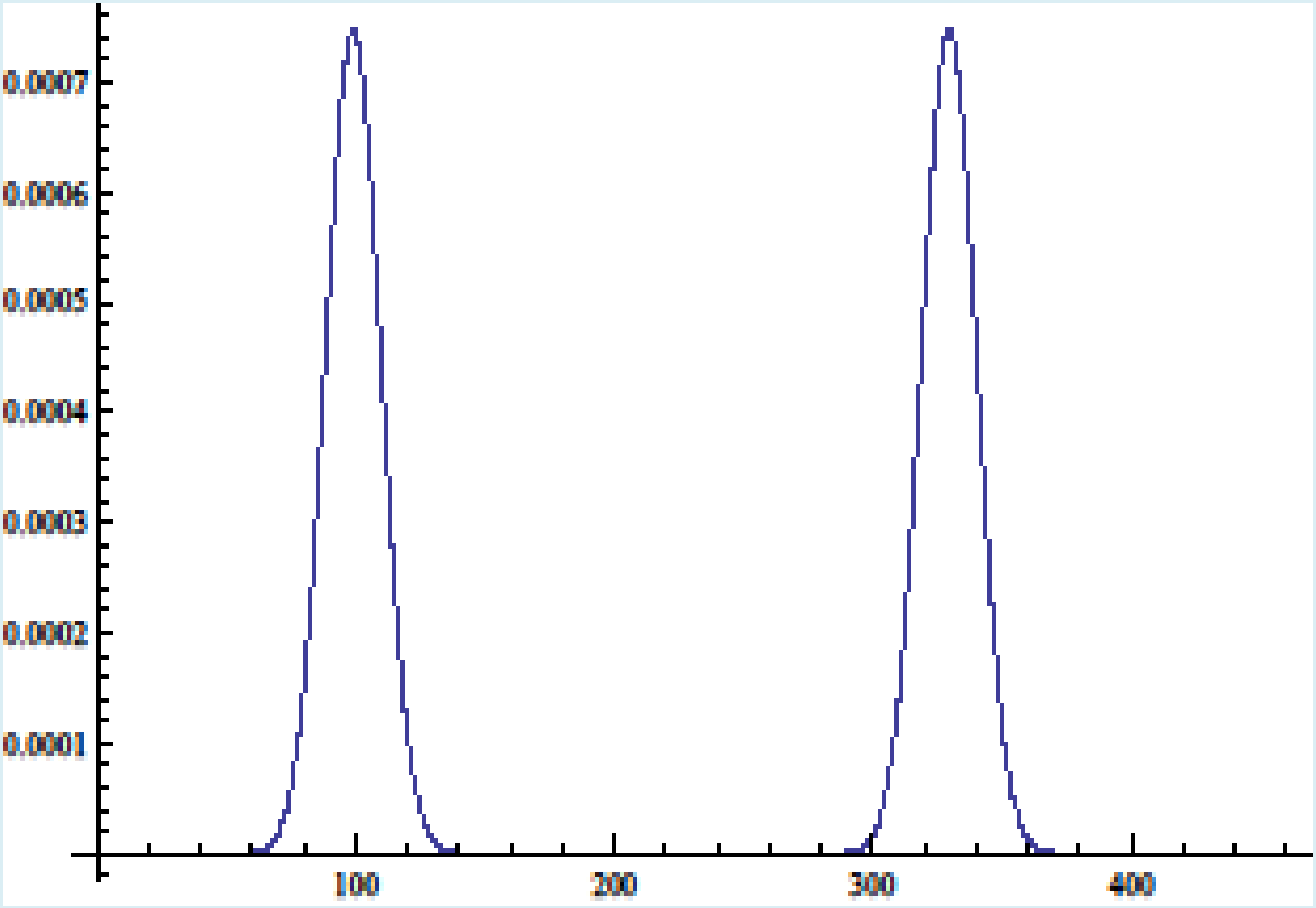
- GROUND STATE PHASE DF



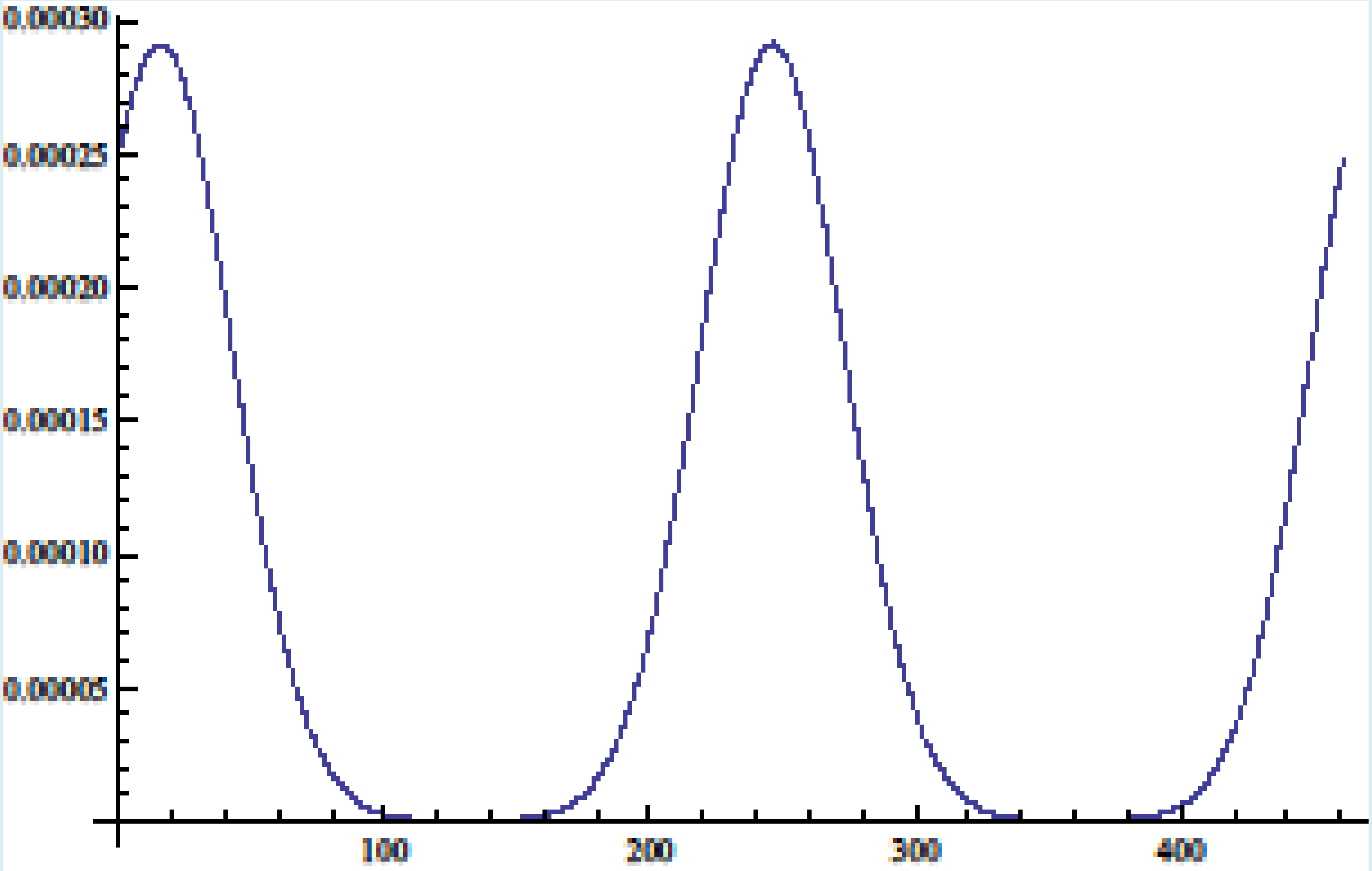
- SCATTERING EXPERIMENT, N=p=461
- GAUSSIAN WF AT T=0



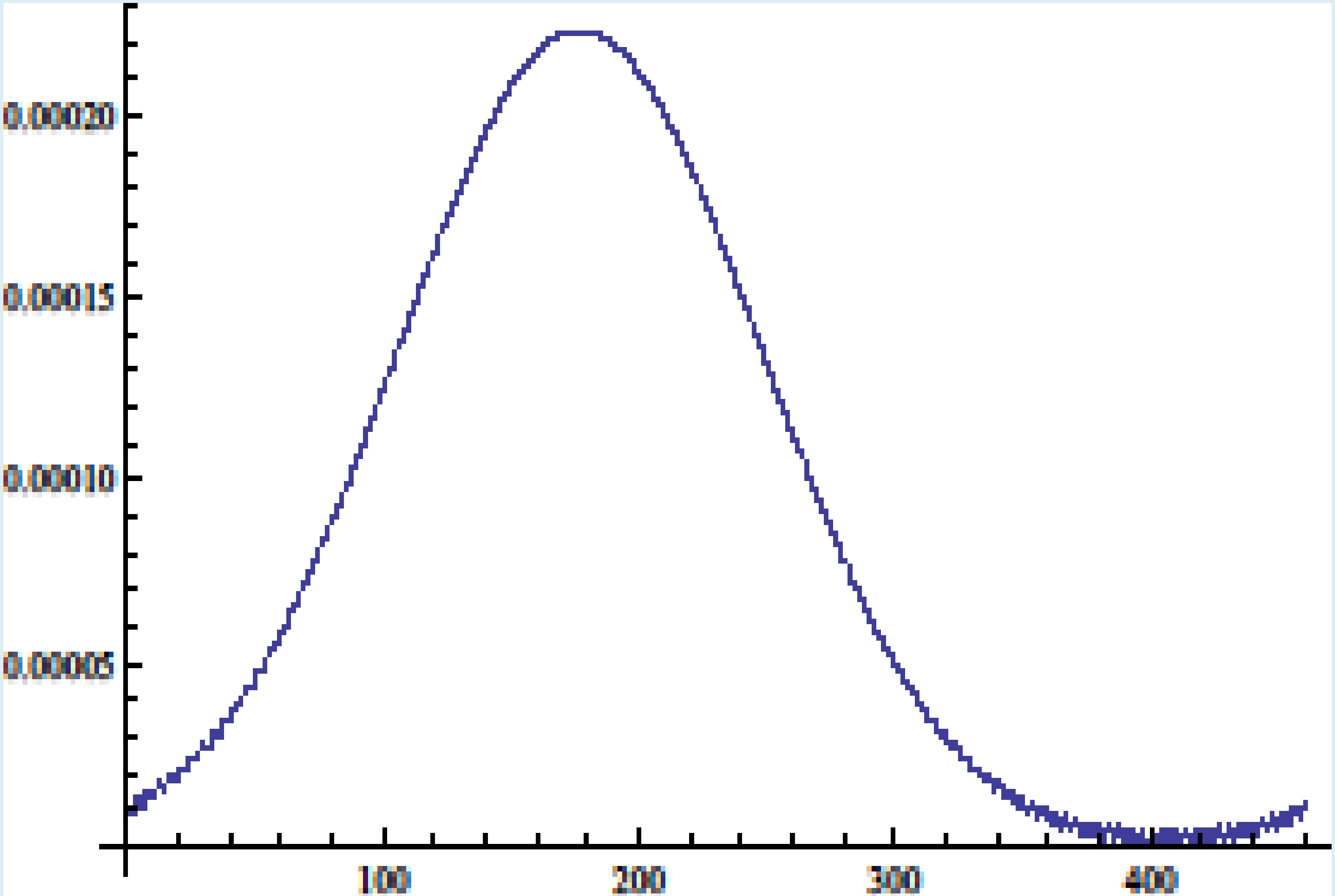
• TIME=1



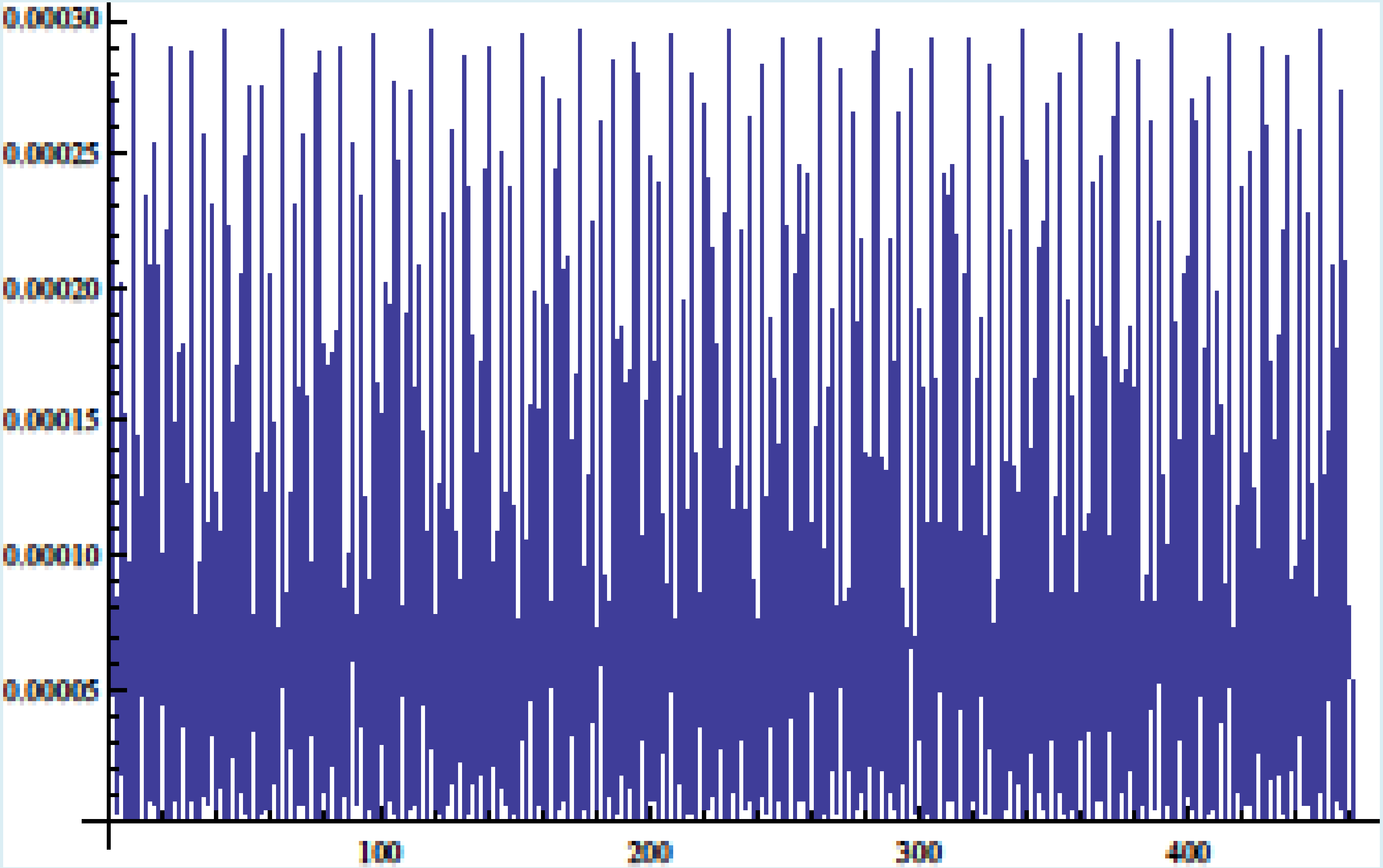
- T=3



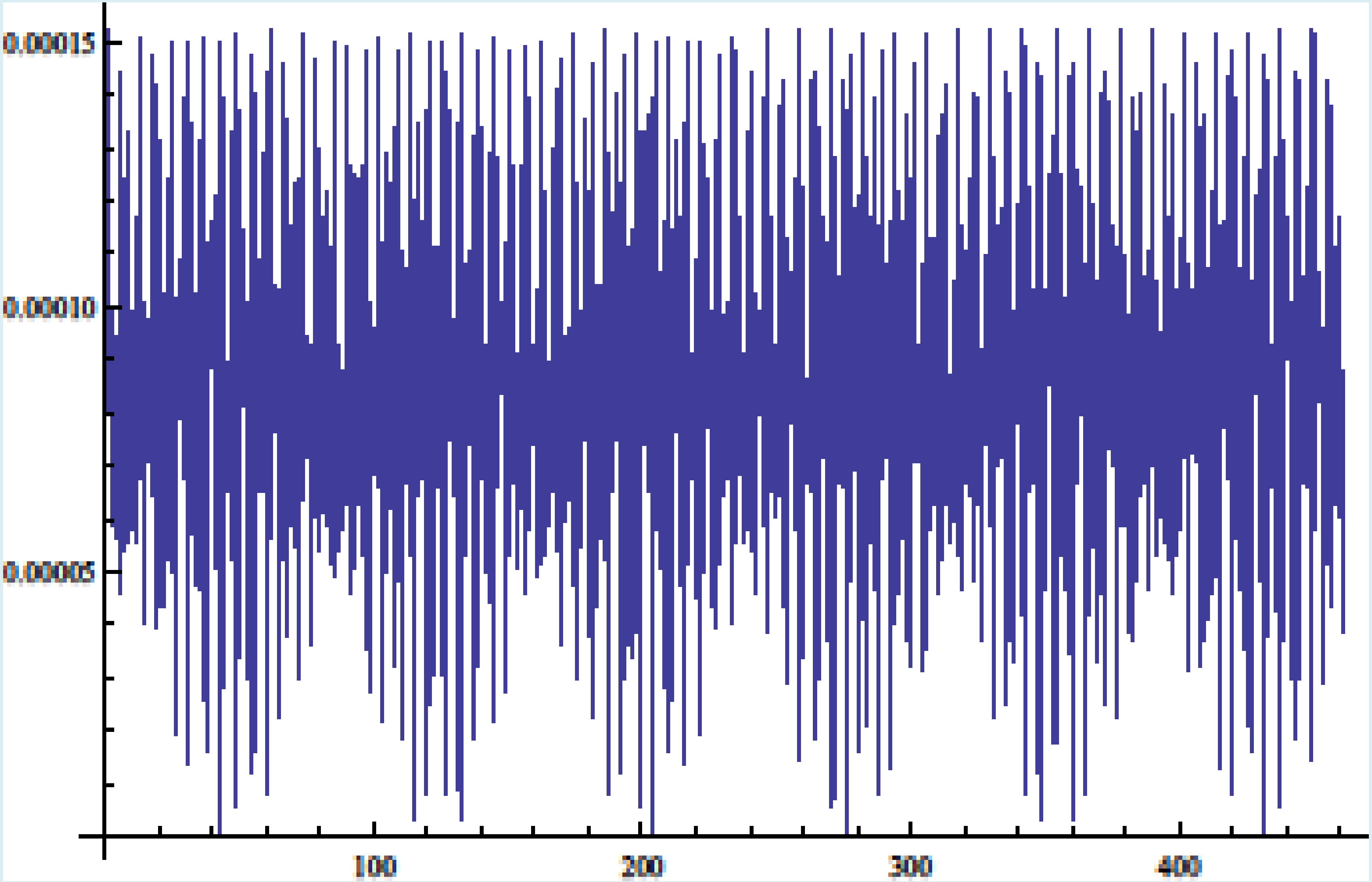
- T=3



- T=4


0.000030
0.000025
0.000020
0.000015
0.000010
0.000005
100
200
300
400

- T=6



- T=7