

Thoughts on a Hamiltonian formulation of sigma models, doubling and possible susy

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Lagrangian

2d (1, 1) superspace

$$\int d^2\xi d\theta^+ d\theta^- \mathcal{L} = \int d^2\xi d\theta^+ d\theta^- (D_+\phi^i E_{ij}(\phi) D_-\phi^j)$$

$$E_{ij} = (E_{(ij)} + E_{[ij]}) := G_{ij} + B_{ij}$$

$$\nabla_+^{(+)} D_- \phi^i = 0 ,$$

$$S_{\pm i} := \frac{\partial \mathcal{L}}{\partial D_{\mp} \phi^i} .$$

$$S_{+i} = -D_+ \phi^j E_{ji}$$

$$S_{-i} = E_{ij} D_- \phi^j .$$

$$\mathcal{H} = [S_{(+i} D_{-)} \phi^i + \mathcal{L}]_|$$

$$\mathcal{H} = S_{+i} E^{ij} S_{-j}$$

$$E^{ij} E_{jk} = \delta_k^i$$

Canonical Equations

$$D_+ \phi^i = \frac{\partial \mathcal{H}}{\partial S_-}$$

$$D_- \phi^i = \frac{\partial \mathcal{H}}{\partial S_+}$$

$$D_{(+S_-)i} = \frac{\partial \mathcal{H}}{\partial \phi^i} .$$

This system is an equivalent formulation of the motion.

A Phase Space Action

First order Lagrangian

$$\mathcal{L}_1 = S_{(+i} D_{-)} \phi^i - S_{+i} E^{ij}(\phi) S_{-j} ,$$

Can be obtained by gauge fixing a model on $\mathbb{T} \oplus \mathbb{T}^*$

$$\mathcal{D}_{(+} \tilde{\phi}_i D_{-)} \phi^i - \mathcal{D}_{+} \tilde{\phi}_i E^{ij}(\phi) \mathcal{D}_{-} \tilde{\phi}_j ,$$

$$\mathcal{D}_{\pm} \tilde{\phi}_i := D_{\pm} \tilde{\phi}_i + S_{\pm i} ,$$

$$\delta S_{\pm i} = D_{\pm} \Lambda_i .$$

The topological term is necessary for equivalence.

The Doubled Model

A $(1, 1)$ Lagrangian on $\mathbb{T} \oplus \mathbb{T}^*$ subject to constraints. In a polarisation:

$$D_{(+}\tilde{\phi}_i D_{-)}\phi^j - D_+\tilde{\phi}_i E^{ij}(\phi) D_- \tilde{\phi}_j ,$$

$$D_+ \tilde{\phi}_i = -D_+ \phi^j E_{ji}$$

$$D_- \tilde{\phi}_i = E_{ij} D_- \phi^j.$$

Inserting the constraints returns the original action on \mathbb{T} . Equivalently this action is obtained via the gauging procedure.

$$D_{(+}\tilde{\phi}_i D_{-)}\phi^i - D_+\tilde{\phi}_i E^{ij}(\phi) D_-\tilde{\phi}_j + \mu D_+ \phi^i E_{ij}(\phi) D_- \phi^j .$$

Additional Supersymmetry

- Can we find additional susy for the Doubled Lagrangian
 $\mathcal{L}(\tilde{\phi}, \phi)$?
c.f. S.Driezen, A. Sevrin, and D.C. Thompson 2016
- Can we find additional susy for the first order Lagrangian
 $\mathcal{L}(\phi, S_{\pm})$?
U.L. 2004, U.L., R.Minasian, A.Tomasiello and M.Zabzine 2005
- Can we find additional susy for the gauged Lagrangian
 $\mathcal{L}(\tilde{\phi}, \phi, S_{\pm})$?

A general (1, 1) model

$$\frac{1}{2} D_+ \phi^A E_{AB} D_- \phi^B$$

has additional susy if

$$\delta\phi^A = \epsilon^+ J_{(+)}^A B D_+ \phi^B + \epsilon^- J_{(-)}^A B D_- \phi^B$$

$$J_{(\pm)}^2 = -\mathbb{1}, \quad J^t G J = G$$

$$\nabla^{(\pm)} J_{(\pm)} = 0.$$

But the constraints?

One (complex) chiral superfield, Kähler potential $K(\phi, \bar{\phi})$

$$\mathcal{L} = D_+ \bar{\phi} K_{,\phi\bar{\phi}} D_- \phi + c.c.$$

Second susy $\phi := (\phi, \bar{\phi})^t$

$$\delta\phi := \delta \begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix} = \epsilon^+ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} D_+ \begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix} =: \epsilon^+ J D_+ \phi$$

$$\tilde{\phi} = (\tilde{\phi}, \bar{\tilde{\phi}})^t$$

$$2D_{(+}\tilde{\phi}^t D_{-)}\phi - D_+\tilde{\phi}^t K^{\bar{\phi}\phi} D_-\tilde{\phi} + 2\mu D_+\phi K_{,\phi\bar{\phi}} D_-\bar{\phi}. ,$$

$$D_+ \bar{\tilde{\phi}} = - D_+ \phi K_{,\phi\bar{\phi}}$$

$$D_- \tilde{\phi} = K_{,\phi\bar{\phi}} D_- \bar{\phi}$$

Not invariant under the unique second susy:

$$\delta\phi = \epsilon^+ J D_+ \phi$$

$$\delta\tilde{\phi} = \pm \epsilon^+ J D_+ \tilde{\phi}$$

If in addition

$$\delta \bar{S}_+ = -i\epsilon^+ D_+ \bar{S}_+ + 2i\epsilon^+ K_{,\phi\bar{\phi}\bar{\phi}} D_+ \phi D_+ \bar{\phi}$$

$$\delta S_- = -i\epsilon^+ D_+ S_- - 2i\epsilon^+ (InK_{,\phi\bar{\phi}})_\phi S_- D_+ \phi$$

then

$$2\mathcal{D}_{(+}\tilde{\phi}^t \mathcal{D}_{-)}\phi - \mathcal{D}_+\tilde{\phi}^t K^{\bar{\phi}\phi} \mathcal{D}_-\tilde{\phi} + 2\mu D_+ \phi K_{,\phi\bar{\phi}} D_- \bar{\phi}.$$

is (off-shell) supersymmetric. So implementation via gauging is supersymmetric.

Generalised

A nontrivial b transformation:

$$\begin{aligned}\mathcal{G} &= (\mathbf{e}^{-b})^t \mathbb{G} \mathbf{e}^{-b} \\ &= (\mathbf{e}^{-b})^t \begin{pmatrix} \mu g & 0 \\ 0 & -g^{-1} \end{pmatrix} \mathbf{e}^{-b} \\ &= \begin{pmatrix} \mu g + bg^{-1}b & -bg^{-1} \\ g^{-1}b & -g^{-1} \end{pmatrix}.\end{aligned}$$

The b transform of the whole Lagrangian is then

$$\begin{aligned}&(\mathcal{D}_+ \phi, \mathcal{D}_+ \tilde{\phi})^t \mathbb{G} (\mathcal{D}_- \phi, \mathcal{D}_- \tilde{\phi}) \\ &\rightarrow (\mathcal{D}_+ \phi, -b \mathcal{D}_+ \phi + \mathcal{D}_+ \tilde{\phi})^t \mathcal{G} (\mathcal{D}_- \phi, -b \mathcal{D}_- \phi + \mathcal{D}_- \tilde{\phi})\end{aligned}$$

On the constraints

$$D_+ \tilde{\phi} = -D_+ \phi g$$
$$D_- \tilde{\phi} = g D_- \phi ,$$

and the Lagrangian becomes

$$(D_+ \phi, D_+ \tilde{\phi})^t \mathbb{G} (D_- \phi, D_- \tilde{\phi})$$
$$\rightarrow (D_+ \phi, -e D_+ \phi)^t \mathcal{G} (D_- \phi, +e^t D_- \phi) ,$$

i.e., the metric \mathbb{G} , gets replaced by \mathcal{G} , and g gets replaced by e and e^t in the constraints.

Evaluating the corresponding action for $\mu = -1$ yields

$$D_+ \phi (g + b) D_- \phi = D_+ \phi e D_- \phi$$

More Generalised

$$\mathcal{G} = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix}.$$

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\mathcal{S} = \eta^{-1}\mathcal{G} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}$$

$$\mathcal{S}^2 = \mathbb{1}$$

$$\mathcal{P}_{(\pm)} := \tfrac{1}{2}(1 \pm \mathcal{S})$$

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{aligned}\mathcal{I} &:= -\Omega^{-1}\mathcal{G}, \quad \mathcal{K} := \Omega^{-1}\eta \\ -\mathcal{I}^2 &= \mathcal{S}^2 = \mathcal{K}^2 = \mathbb{1} \\ \mathcal{ISK} &= -\mathbb{1}, \quad \{\mathcal{I}, \mathcal{S}\} = \{\mathcal{S}, \mathcal{K}\} = \{\mathcal{K}, \mathcal{I}\} = 0\end{aligned}$$

$$\Phi^M = \begin{pmatrix} \phi^\mu \\ \tilde{\phi}_{\tilde{\mu}} \end{pmatrix}$$

$$S_2 = \frac{1}{4} \int d^2x d^2\theta (D_+ \Phi)^t \mathcal{K} \mathcal{G} D_- \Phi ,$$

$$\mathcal{P}_{(\pm)} D_{\pm} \Phi = 0 , \quad \Rightarrow D_+ \tilde{\phi}_{\tilde{\mu}} = - D_+ \phi^\mu e_{\mu \tilde{\mu}} , \quad D_- \tilde{\phi}_{\tilde{\mu}} = e_{\tilde{\mu} \mu} D_- \phi^\mu$$

$$S_2 \rightarrow \frac{1}{2} \int d^2x d^2\theta D_+ \phi e D_- \phi$$

or

$$\begin{aligned} S_1 &= \frac{1}{2} \int d^2x d^2\theta D_+ \phi \mathcal{G} D_- \phi - D_+ (\tilde{\phi} D_-) \phi \\ &\rightarrow \frac{1}{2} \int d^2x d^2\theta D_+ \phi e D_- \phi \end{aligned} \tag{0.1}$$

$$\mathcal{J}^2 = -\mathbb{1} , \quad \mathcal{J}^t \mathcal{G} \mathcal{J} = \mathcal{G} , \quad \mathcal{J}^t \eta \mathcal{J} = \eta$$

$$[\mathcal{J}, \mathcal{S}] = 0 , \quad [\mathcal{J}, \Omega] = 0 ,$$

$$\Rightarrow \mathcal{J} = e^b \mathcal{J}_0 e^{-b}$$

$$\mathcal{J}_0 = \left(\begin{array}{cc} J & 0 \\ 0 & -J^t \end{array} \right) .$$

$$\delta \Phi = \epsilon \mathcal{J} D \Phi$$