Unitary Orbits in Quantum Field Theory

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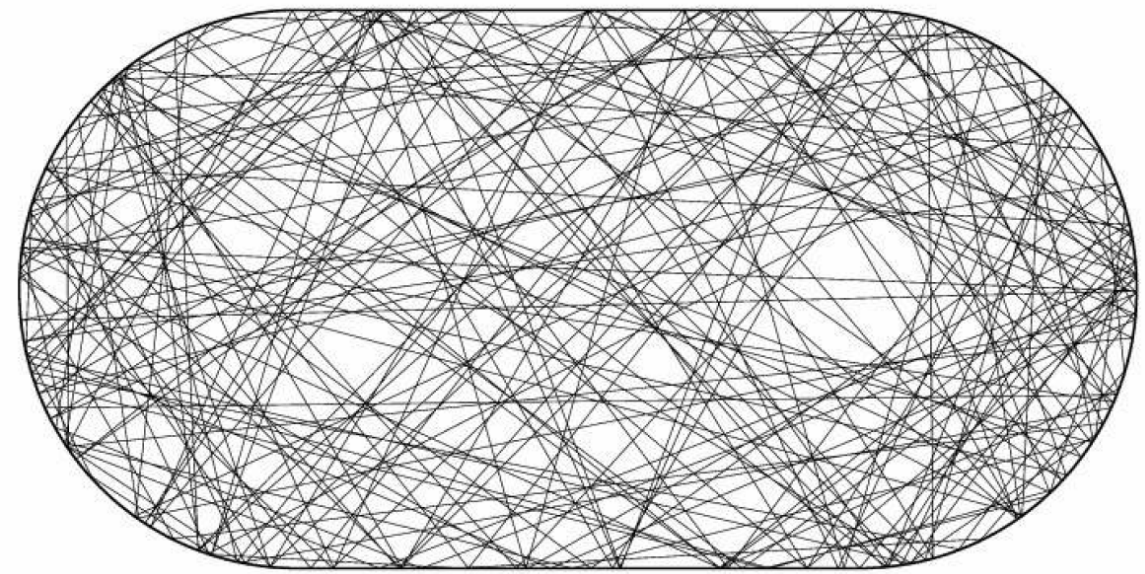
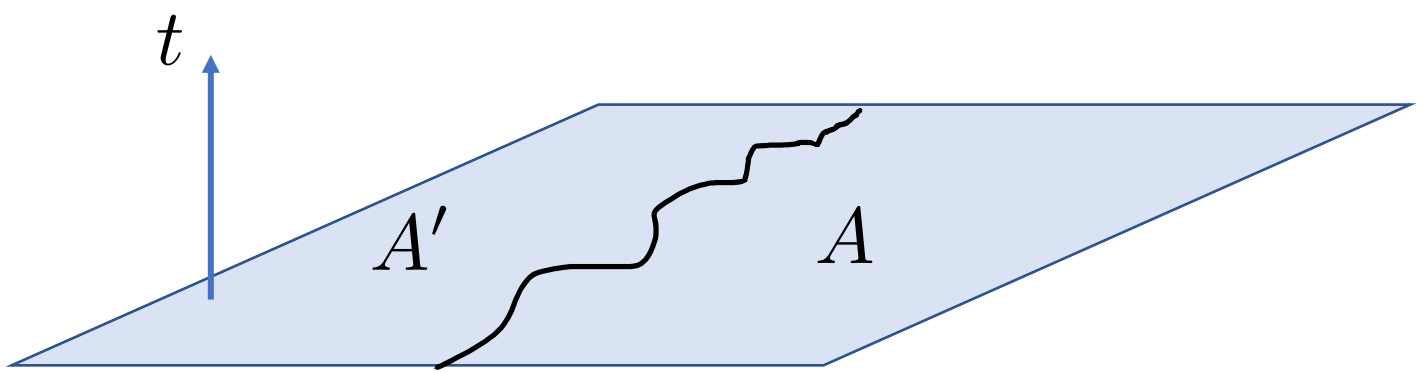


- Quantum fields are not finite quantum systems. Entanglement theory in QFT is different from that of finite systems.
- In QFT there exist no Renyi entropies, no entanglement entropy, nor any unitarily invariant measure of a local state.
- In QFT the unitary orbit of local unitaries is dense in the Hilbert space! (Connes, Stormer 1978)



$$\inf_{U\in\mathcal{A},U'\in\mathcal{A}'} \| |\Psi\rangle - UU'|\Omega\rangle \| = 0 \ .$$

Why does it matter? Entanglement theory

- Entanglement is invariant under local unitaries.

$$|00\rangle + |11\rangle \to |++\rangle + |--\rangle \ . \qquad |\Psi\rangle \to (U \otimes U')|\Psi\rangle$$

Dense unitary orbits means that all state have the same amount of entanglement!

- Information content of a density matrix is a spectrum and a choice of basis (unitary)

$$\rho_A = \sum_k p_k |k\rangle\langle k|$$

spectrum

Choice of basis

- Measures of entanglement are functions of the spectrum (unitary invariant)

$$\rho \to U\rho U^\dagger, \qquad \sum_k p_k^n = \mathrm{tr}(\rho^n),$$

$$|\Psi\rangle \simeq UU'|\Omega\rangle$$

- In QFT unitary orbits are dense, therefore, there are no Renyi entropies, entanglement entropies, or any unitary invariant measure of a single state.
- All that exists is the "relative entropy" of two states. The starting point of the entanglement theory of quantum fields. (Araki 1976)

Local states: finite quantum systems

- The local state is described by a density matrix. linear map

$$\omega : \mathcal{A}_A \to \mathbb{C}, \omega(a) = \mathrm{tr}(\rho a)$$

- Entanglement: $0 < p_k < 1$

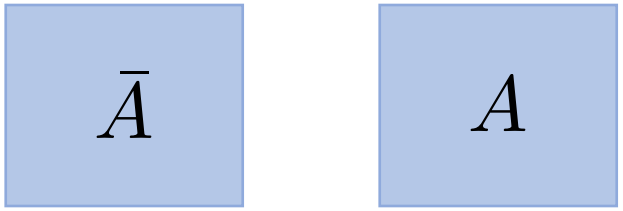
Acting on A you can excite $\bar{A}$

If all $0 < p_k$ acting on A you can reach all excitations. (Reeh-Schlieder property of a state)

- Spectrum is totally discrete: there are "recurrences"

Purification as a global vector

$$|\Omega\rangle = \sum_k \sqrt{p_k} |k\rangle_A \otimes |k\rangle_{\bar{A}}$$



$$\rho_A = \sum_k p_k |k\rangle\langle k|$$

Local states: quantum field theory

- The local state is $\omega(a) = \langle\Omega|a|\Omega\rangle$

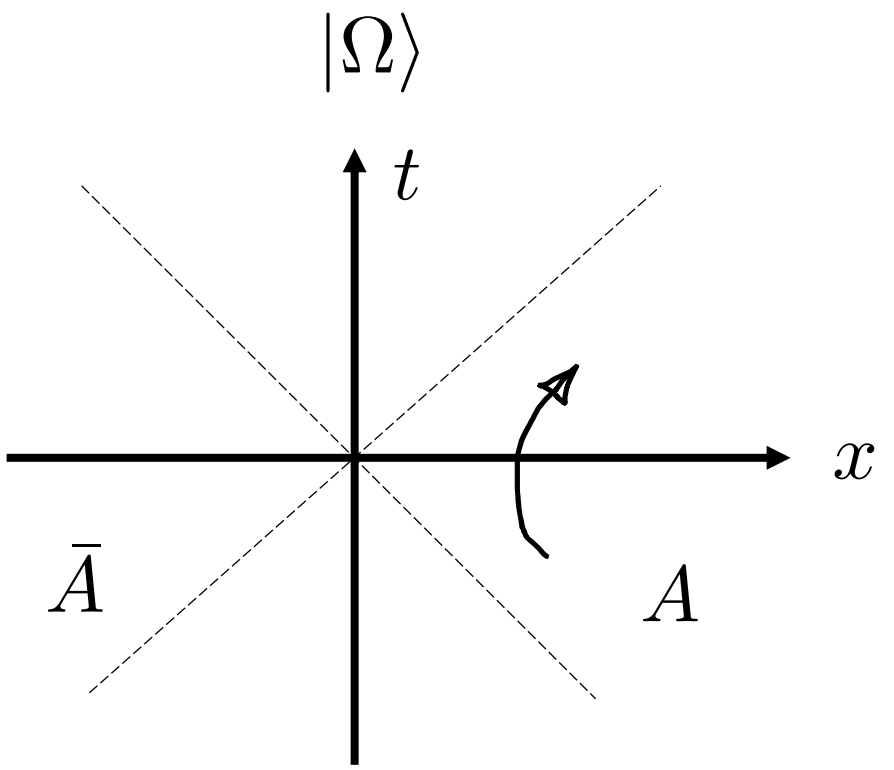
- "Totally entangled":

Acting on A you can reach all excitations

$$\overline{\mathcal{A}|\Omega\rangle} = \mathcal{H}$$

Vacuum has the Reeh-Schlieder property.

- Spectrum is continuous: there are no eigenvalues: the only normalizable boost invariant state of QFT is the vacuum.



$$\rho_A \sim e^{-2\pi K_x}$$

Local state as a dynamical system

- Density matrix is a positive operator $\rho = e^{-H_\rho}$

  "Modualr Hamiltonian" introduces dynamics $\rho^{it} = e^{-itH_\rho}$

- The spectrum of the density matrix is reflected in the spectrum of the modular Hamiltonian.

- Whether it is totally discrete or a continuum is diagnosed by looking at long time evolution of operators under "Modular flow"

  If the spectrum is totally discrete there are recurrences.
  If the spectrum is totally continuous the flow is ergodic.

Modular flow of purified states

- Modular evolution: $a \in \mathcal{A}: \qquad a(t) = \rho^{it} a \rho^{-it}$

Modular operator $\qquad \Delta_\rho = \rho \otimes \rho^{-1}$

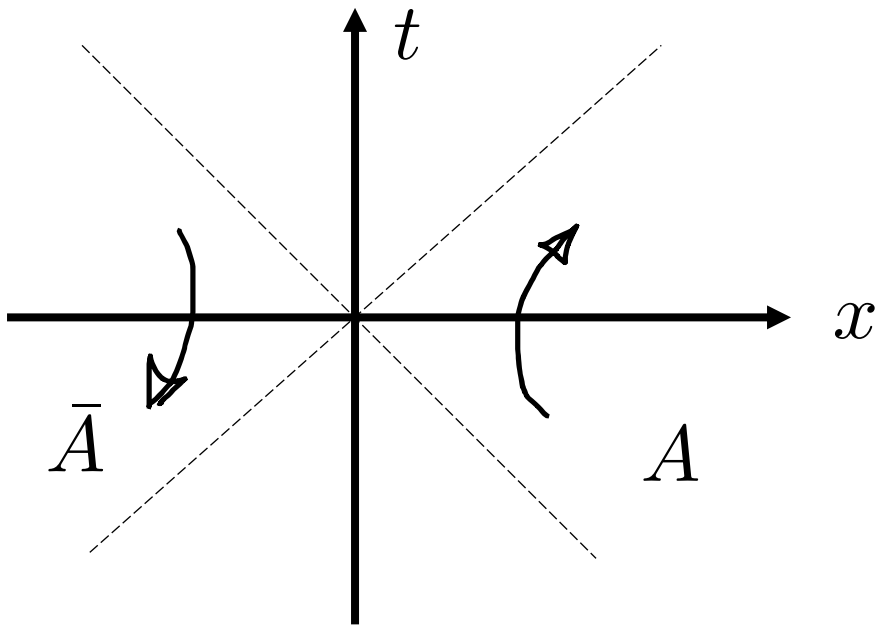
Hamiltonian $\qquad H_\rho = -\log \Delta_\rho = -\log \rho \otimes \mathbb{I} + \mathbb{I} \otimes \log \rho$

$$|\Psi\rangle = \sum_{kk'} \Psi_{kk'} |k, k'\rangle, \qquad e^{itH}|\Psi\rangle = \sum_{kk'} \Psi_{kk'} e^{it \log(p_k/p_{k'})} |k, k'\rangle$$

- If the spectrum is discrete (there are eigenvalues) there are no recurrences.

$$t \sim 1/\log(p_k/p_{k'})$$

- One way to think of ergodic flow is in terms of a totally discrete spectrum in the limit where the set of eigenvalue gaps $\{p_k/p_{k'}\}$ becomes dense in positive numbers. The state never comes back where it started (ergodicity)
- If the spectrum is continuous (no eigenvalues) the evolution is ergodic.



$$\rho_A \sim e^{-2\pi K_x}$$

$$H = K_x \otimes \mathbb{I} + \mathbb{I} \otimes K_x = \hat{K}_x$$

$$\lim_{t\to\infty} \langle \Phi | e^{it\hat{K}_x} | \Psi \rangle = \langle \Phi | \Omega \rangle \ .$$

At large time, all excitations are boosted away and the state looks like vacuum. (Ergodicity)

How close are two local quantum states?

$$d(\rho, \omega) = \mathrm{tr}((\sqrt{\rho} - \sqrt{\omega})^2) = 2(1 - \mathrm{tr}(\sqrt{\rho}\sqrt{\omega}))$$

- Equivalence class under unitary orbits $\rho \sim U\rho U^\dagger$
- Distance between unitary orbits

$$\inf_U d(\rho, U\omega U^\dagger) = \inf_U \mathrm{tr}((\sqrt{\rho} - U\sqrt{\omega}U^\dagger)^2)$$

$$\leq \sum_k (\sqrt{p_k} - \sqrt{q_k})^2$$

- Relabeling unitaries $$p_k \to p_{\sigma(k)}$$
- Distance between unitary orbits

$$\inf_U \mathrm{tr}((\sqrt{\rho} - U\sqrt{\omega}U^\dagger)^2) \leq \inf_{\sigma \in S_d} \sum_k (\sqrt{p_k} - \sqrt{q_{\sigma(k)}})^2$$

- Distance as a commutator: Introduce an auxiliary qubit degree of freedom and consider the density matrix

$$\theta = \rho \otimes |0\rangle\langle 0| + \omega \otimes |1\rangle\langle 1|$$

and the partial isometry

$$V = U \otimes |0\rangle\langle 1|$$

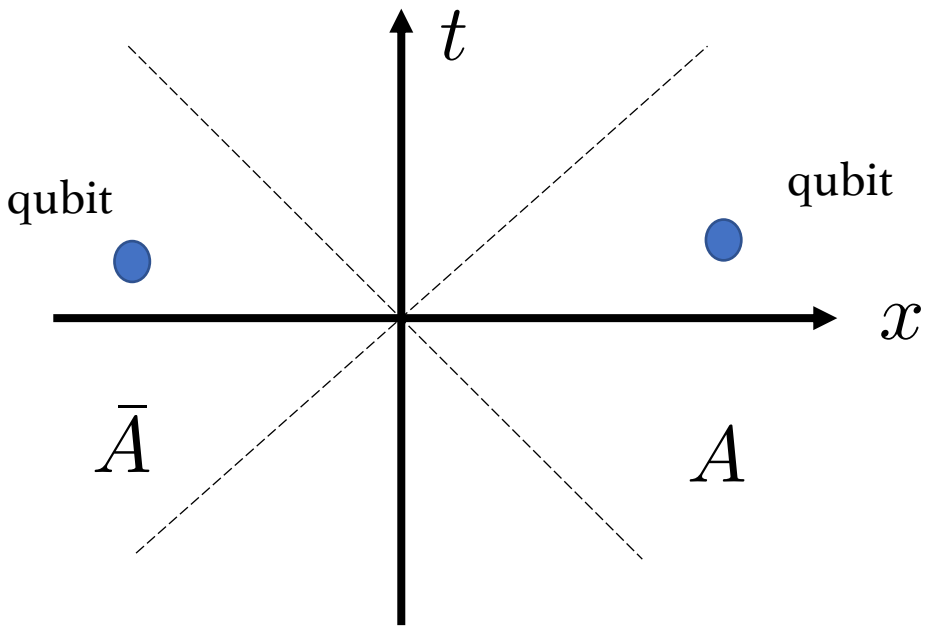
then

$$\mathrm{tr}((\sqrt{\rho} - \sqrt{\omega})^2) = \mathrm{tr}\left(|[V, \sqrt{\theta}]|^2\right)$$

What is the unitary that relates two states?

$$|\Psi\rangle,\ |\Omega\rangle$$

$$|\Theta\rangle = |\Omega\rangle \otimes |00\rangle + |\Psi\rangle \otimes |11\rangle$$



- We are looking for operators that commute with θ

modular flow in this state
a unitary "cocyle"

$$u(t) = \theta^{it}(1 \otimes |0\rangle\langle 1|)\theta^{-it} \sim \rho^{it}\omega^{-it}$$

- Then,

$$\int_{-\infty}^{\infty} dt\, [u(t), \sqrt{\theta}] = 0$$

$$\int_{-\infty}^{\infty} dt\, \theta^{it} (1 \otimes |0\rangle\langle 1|) \theta^{-it}$$

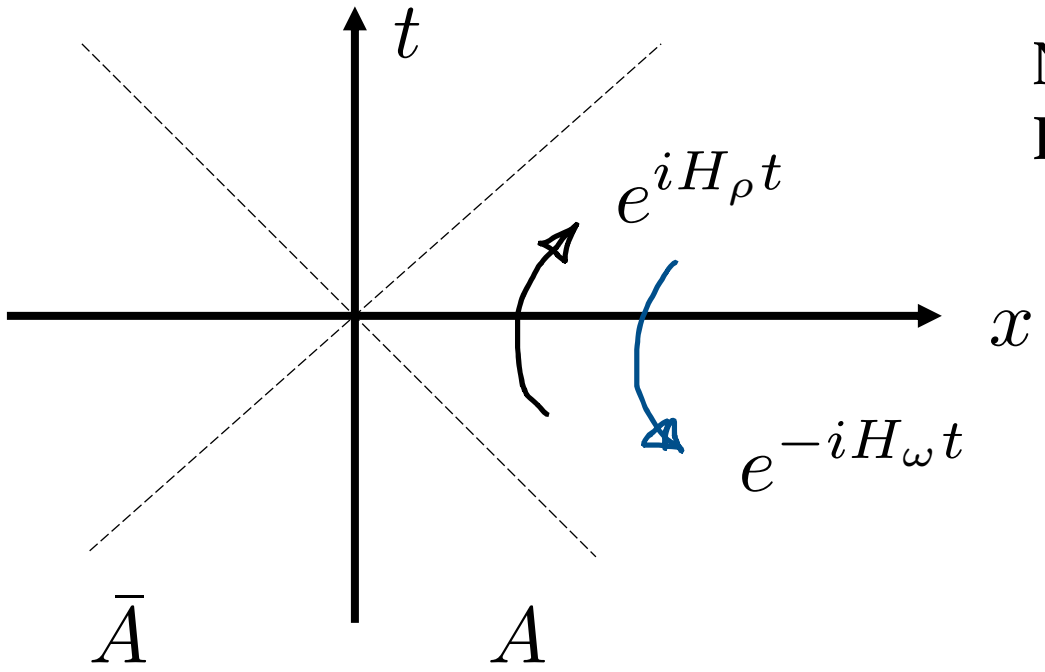
however, this is not a unitary. To find the unitary, one has look at the spectral decomposition of this operator. However, there is a shortcut:

Weak limit: if the spectrum of the cocycle is continuous then

$$w.o. \lim_{t \to \infty} [u(t), \sqrt{\theta}] = 0 \ .$$

(comment: this operator plays a central role in the proof of QNEC) (Ceyhan, Faulkner 2018)

- The physical interpretation of the cocycle:



Near the edge of A it acts like identity.
Far at infinity it acts like identity

- **Relative Entropy:**

$$S(\Psi\|\Omega) = i\frac{d}{dt}\langle\Psi|u_{\Omega|\Psi}(t)\Psi\rangle$$

Summary:

There exist unitaries U and U' such that for any two states

$$|\Psi\rangle \simeq UU'|\Omega\rangle$$

Furthermore,

$$\forall|\Phi\rangle, \qquad \lim_{t\to\infty} |\langle\Phi|u(t)u'(t)\Psi\rangle - \langle\Phi|\Omega\rangle| = 0 \ .$$

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- Cocycle and relative entropy are at the core of the entanglement theory of quantum fields.

Thank you!