Lorentzian CFT correlators in momentum space

Hadi Godazgar

Albert Einstein Institute
Potsdam

with Teresa Bautista

arXiv:1908.04733 [hep-th]

September, 16 2019

Tenth Crete Regional Meeting on String Theory

Kolymbari

Introduction

- CFT correlators in position space well-understood [Polyakov '70; Osborn Petkou '94]
- 2-point and 3-point functions fixed by conformal symmetry and given by relative distances
- Euclidean simple: symmetric under permutations and analytic at non-coincident points
- Lorentzian correlators obtained straightforwardly by Wick rotation using the $i\epsilon$-prescription
- Simpler understanding of OPE in position space (convergence)

Why Lorentzian and momentum space?

Lorentzian more complicated but also richer; encode causality

- E.g. causality proof of ANEC [Hartman Kundu Tajdini '16]
- E.g. Analytic bootstrap [Fitzpatrick Kaplan Poland Simmons-Duffin '12][Komargodski Zhiboedov '12]...
- E.g. Lorentzian inversion formula [Caron-Huot '17]...

Momentum space

- Cosmology [Maldacena Pimental '11][Creminelli, Norena, Simonovic '12]...[Sleight Taronna '19]
- Want to understand collider bounds as expectation values [Hofman Maldacena '08]...
- Also more generally light-ray operators [Zhiboedov...]

Even in Euclidean signature correlators in momentum space have only been found relatively recently [Bzowski McFadden Skenderis '13–'18][Coriano Delle Rose Mottola Serino '13]...

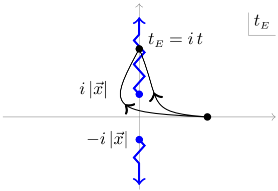
Outline

- 2-point function
- 3-point function
- $\langle \mathcal{O} T \mathcal{O} \rangle$ correlator
- Application to ANEC and collider bounds
- Outlook

2-point function and main idea

Euclidean 2-point function of scalar operators $\mathcal{O}_1$ and $\mathcal{O}_2$ with dimensions Δ

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\rangle_E = \frac{1}{(t_E^2 + |\vec{x}|^2)^\Delta}$$



The two different Wick rotations $t_E = it$

$\longrightarrow$ two possible Wightman 2-point functions

$\longrightarrow$ Wick rotations given by $i\epsilon$ prescription

Lorentzian correlators

Order of operators dictated by $i\epsilon$ prescription ($\epsilon > 0$)

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\rangle = \frac{1}{\left(-(t - i\,\epsilon)^2 + |\vec{x}|^2\right)^{\Delta}}$$

$$\langle \mathcal{O}_2(0)\mathcal{O}_1(x)\rangle = \frac{1}{\left(-(t + i\,\epsilon)^2 + |\vec{x}|^2\right)^{\Delta}}$$

- Relates unitary Lorentzian theory to reflection-positive Euclidean theory
- Prescription such that for ordering

$$\ldots e^{iH(t_i - i\epsilon_i)}\,\mathcal{O}_i(\vec{x}_i)\,e^{-iH\left((t_i - t_j) - i(\epsilon_i - \epsilon_j)\right)}\,\mathcal{O}_j(\vec{x}_j)\,e^{-iH(t_j - i\epsilon_j)}\ldots$$

$\epsilon_i > \epsilon_j$ to have a decaying exponential

2-point function in momentum space

- Fourier transform of Euclidean correlator (d-dimensions)

$$\langle\!\langle \mathcal{O}(p)\mathcal{O}(-p)\rangle\!\rangle_E = \frac{\pi^{d/2}\,\Gamma(d/2-\Delta)}{2^{2\Delta-d}\,\Gamma(\Delta)}\,\left(p_E^2+|\vec{p}|^2\right)^{\Delta-d/2}$$

- Fourier transform of Lorentzian correlator

$$\langle\!\langle \mathcal{O}(p)\mathcal{O}(-p)\rangle\!\rangle = \frac{\pi^{d/2+1}}{2^{2\Delta-d-1}\,\Gamma(\Delta-\frac{d}{2}+1)\,\Gamma(\Delta)}\,\theta(p^0-|\vec{p}|)\,|p|^{2\Delta-d}\,,$$

$$|p| = \sqrt{(p^0)^2-|\vec{p}|^2}$$

- Not straightforward $p_E = -ip^0$
- E.g. Lorentzian correlator well-defined at $\Delta - d/2 \in \mathbb{N}^0$
- Direct Fourier transform not simple
  $\longrightarrow$ even more complicated for 3-point functions!

Wick rotation in momentum space

- By definition, correlator in Euclidean position and momentum space related by Fourier transform

$$G_E(x) = \int \frac{d^d p}{(2\pi)^d}\, e^{ip\cdot x}\, G_E(p)\,,$$

- Simple relation between Euclidean and Lorenztian correlators in momentum space

$$G(x) = G_E(i(t-i\epsilon), \vec{x})$$

- Hence

$$G(x) = \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\, e^{i\vec{p}\cdot\vec{x}} \int_{-\infty}^{\infty} \frac{dp_E}{2\pi}\, e^{-p_E(t-i\epsilon)}\, G_E(p_E, \vec{p})$$

- No longer a Fourier transform, but turn into a Fourier tranform by Wick-rotating p_E to the imaginary axis
$\longrightarrow$ yields Lorentzian correlator in momentum space

Example: 2-point function

Euclidean 2-point function in momentum space

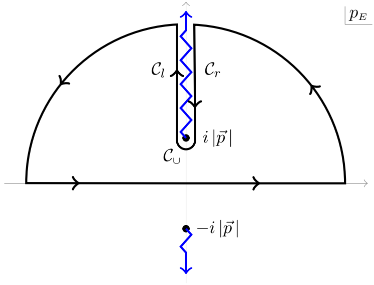
$G_E(p) \sim (p_E^2 + |\vec{p}|^2)^{\Delta - d/2}$

Branch points at $\pm i|\vec{p}|$

Deform contour to integrate along imaginary axis

Contribution from arcs vanish because of ϵ, $e^{-p_E(t-i\epsilon)}$

Integral around branch cut gives the step function $\theta(p^0 - |\vec{p}\,|)$



Deforming contour gives Lorentzian 2-point function in momentum space without need to Fourier transform

3-point function

Euclidean 3-point functions in momentum space [Bzowski McFadden Skenderis '13–'18]

$$C_E(\{\beta_j\}) \sim \prod_{j=1}^{3} \frac{\Gamma(d/2-\beta_j)}{\Gamma(\beta_j)} \int \frac{d^d k}{(2\pi)^d} \frac{1}{|p_2+k|^{d-2\beta_1}\,|p_1-k|^{d-2\beta_2}\,|k|^{d-2\beta_3}}$$

$$C_E(\{\beta_j\}) \sim \int_0^\infty dt\, t^{d/2-1} \prod_{j=1}^{3} |p_j|^{\nu_j}\, K_{\nu_j}(|p_j|\, t)$$

$$\beta_j = \frac{\Delta_t}{2} - \Delta_j, \qquad \nu_j = \Delta_j - \frac{d}{2}$$

Requires renormalisation for

$$\beta_j - \frac{d}{2} \in \mathbb{N}^0$$

Position space correlator expression

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3)\rangle = \frac{c_{123}}{(x_{23}^2)^{\beta_1}\,(x_{13}^2)^{\beta_2}\,(x_{12}^2)^{\beta_3}}$$

not defined at coincident points

Lorentzian 3-point function in momentum space

$$C(\{\beta_j\}) \sim \int \frac{d^d k}{(2\pi)^d} \, \frac{\theta(k^0 - |\vec{k}|)\, \theta(p_2^0 + k^0 - |\vec{p}_2 + \vec{k}|)\, \theta(p_1^0 - k^0 - |\vec{p}_1 - \vec{k}|)}{|p_2 + k|^{d-2\beta_1}\, |p_1 - k|^{d-2\beta_2}\, |k|^{d-2\beta_3}}$$

- No renomalisation needed because of $i\epsilon$ prescription
  $\longrightarrow$ no coincident points
- 3-point function vanishes when $p_1 = 0$ or $p_3 = 0$

Fourier transform of triple-K expression

$$C(\{\beta_j\}) \sim \int_0^\infty dt\, t^{d/2-1} \prod_{j=1}^{3} p_j^{\,\nu_j}\, \mathcal{B}_{\nu_j}(p_j\, t)$$

- each term is multplied by various combinations of step function, e.g.

$$\theta(p_1^0 - |\vec{p}_1|)\, \theta(p_2^0 - |\vec{p}_2|),$$

$$\theta(p_1^0 - |\vec{p}_1|)\, \theta(-p_3^0 - |\vec{p}_3|)\, \theta(-p_1^0 + |\vec{p}_3|)\, \theta(-p_2^0 + |\vec{p}_2|)$$

- $\mathcal{B}_{\nu_j}$ Bessel functions of the first (J) or second kind (Y) or modified Bessel function K

Tensorial 3-point functions

Tensorial functions found from scalar case by differentiation, e.g.

$$\int \prod_i d^d x_i \, e^{-ip_i \cdot x_i} \frac{x_{12\,\mu} x_{12\,\nu}}{(x_{23}^2)^{\beta_1} \, (x_{13}^2)^{\beta_2} \, (x_{12}^2)^{\beta_3}}$$

$$\sim \delta^{(d)}(p_1 + p_2 + p_3) \left(\frac{\partial}{\partial p_1^\mu} - \frac{\partial}{\partial p_2^\mu} \right) \left(\frac{\partial}{\partial p_1^\nu} - \frac{\partial}{\partial p_2^\nu} \right) C(p_1, p_2; \{\beta_j\})$$

Using Osborn-Petkou, e.g. $\langle\!\langle \mathcal{O}(p_1) T_{--}(p_2) \mathcal{O}(p_3) \rangle\!\rangle$ is

$$\int \frac{d^d k}{(2\pi)^d} \frac{\theta(p_1^0 - k^0 - |\vec{p}_1 - \vec{k}|) \, \delta(k^0 - |\vec{k}|) \, \delta(p_2^0 + k^0 - |\vec{p}_2 + \vec{k}|)}{|\vec{k}| \, |\vec{p}_2 + \vec{k}| \, |p_1 - k|^{2(d-1-\Delta)}}$$

$$\times \left[\frac{4(d-1)}{d-2} \, \left(k^+ k^+ + \, k^+ p_2^+ \right) + \, p_2^+ p_2^+ \right]$$

$$k^+ = k^0 + k^1$$

ANEC expectation values on HM states

- Original calculation done in position space
- Seems more natural in momentum space: ANEC operator commutes with momentum operator at ∞
- Positivity of ANEC operator in HM states:

$$\langle \mathcal{E} \rangle = \lim_{r\to\infty} r^{d-2} \langle \mathcal{O}(q)| \int_{-\infty}^{\infty} dx^- \, T_{--}(x^+, x^-) \, |\mathcal{O}(q)\rangle \geq 0$$

- In momentum space

$$\langle \mathcal{E} \rangle = \lim_{r\to\infty} r^{d-2} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \, e^{2ip^1 r} \langle\!\langle \mathcal{O}(q, \vec{0}\,) \, T_{--}(-p^1, \vec{p}\,) \, \mathcal{O}(p^1 - q, -\vec{p}\,) \rangle\!\rangle$$

- Integrals just reduce to δ-function integrals
- In $r \to \infty$ limit, momenta of the ANEC operator become zero $\longrightarrow$ clear why this is an expectation value

$$\langle \mathcal{E} \rangle \sim \langle\!\langle \mathcal{O}(q, \vec{0}\,) \, T_{--}(0, \vec{0}\,) \, \mathcal{O}(-q, \vec{0}\,) \rangle\!\rangle$$

Outlook

- A way to obtain Lorentzian correlators from Euclidean ones by analytic continuation
- Only need to know analytic properties of the Euclidean correlator in momentum space
- ANEC simplifies considerably
- More recently, expression in terms of Appell F_4 function [Gillioz '19]
  $\longrightarrow$ what is relation?
- Nicer treatment of tensorial correlators
- Can we use ANEC away from fixed point?

Thank you!