Osborn Equation and Irrelevant Operators

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Plan of the Talk

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1. Review of Osborn Equation

Integer dimensional operators in CFT (energy momentum tensors, moduli, ..) give rise to anomalies.

At separate points the correlators are well defined but at coincident points there is a logarithmic ("type B") or powerlike ("type A") singularity which when regulated leads to violations of the conformal Ward identities i.e. to a "Weyl anomaly".

The usual Weyl anomalies related to single logarithms are well understood but e.g. for marginal but not truly marginal operators there are increasing number of logarithms when increasing the number of operators in the correlator.

This situation is covered for marginal and relevant operators by the Osborn Equation ("Local Callan-Symanzik Equation").

The Weyl variation of the generating functional is generalized to :

$$\delta_\sigma W = \int d^d x \left\{ 2\sigma(x) g_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)} + \sum_i \beta_i\big(\{J_j\}, g_{\mu\nu}, \sigma\big) \frac{\delta}{\delta J_i(x)} \right\} W$$

The first term represents the canonical Weyl transformation of the metric and the second term involves the local "beta-functions" of the sources.

The Osborn Equation states that the Weyl variation of W gives the Weyl anomalies

$$\delta_\sigma W = \int d^d x \sqrt{g}\, \sigma(x) \mathcal{A}(\{J_i\}, g)$$

$\mathcal{A}$ being **local .**

There are two cohomological structures underlying the Osborn Equation which are related to the abelian Weyl symmetry

$$[\delta_{\sigma_1}, \delta_{\sigma_2}] = 0$$

a) The transformation of the sources

$$\delta_\sigma g_{\mu\nu} \equiv 2\sigma g_{\mu\nu}$$

$$\delta_\sigma J_i \equiv \beta_i\big(\{J_j\}, g_{\mu\nu}; \sigma\big)$$

should obey the integrability condition:

$$\delta_{\sigma_2}\beta_i\big(\{J_j\}, g_{\mu\nu}; \sigma_1\big) = \delta_{\sigma_1}\beta_i\big(\{J_j\}, g_{\mu\nu}; \sigma_2\big)$$

The integrability conditions have trivial solutions:

$$J_i'(x) \equiv \gamma_i\big(\{J_j\}, g_{\mu\nu}\big)$$

such that $J_i'(x)$ transforms just following the dimension, undoing the beta-function:

$$\delta_\sigma J_i'(x) = \sigma(x)\big(\Delta_i - d\big) J_i'(x)$$

b) For non-trivial beta-functions the Weyl anomaly is the solution of a second cohomology problem: The anomaly should obey

$$\delta_{\sigma_2} \int d^d x \sqrt{g}\, \sigma_1(x) \mathcal{A}\big(\{J_i\}, g_{\mu\nu}\big) = \delta_{\sigma_1} \int d^d x \sqrt{g}\, \sigma_2(x) \mathcal{A}\big(\{J_i\}, g_{\mu\nu}\big)$$

modulo variations of local diffeo-invariant functionals of $\{J_i\}$ and $g_{\mu\nu}$.

2. Two Source Anomalies. No go Theorem

Consider the two-point function of scalar primary operators of dimension Δ:

$$\langle \mathcal{O}(x)\,\mathcal{O}(0)\rangle = \frac{N}{|x|^{2\Delta}}$$

For $\Delta - d/2 = n \in \mathbb{N}_0$ this is singular. Momentum space regularization gives:

$$\langle \mathcal{O}(p)\,\mathcal{O}(-p)\rangle = (-1)^{n+1}\frac{N\,\pi^{d/2}}{2^{2n}\Gamma(n+1)\Gamma(n+\frac{d}{2})}p^{2n}\Big(\log(p^2/\mu^2) + c_{n,d}\Big)$$

In configuration space $p^{2n} \sim \Box^n$ This leads to a Weyl Anomaly with the "classical" transformation rules since to this order the beta-function does not contribute .

The cohomological analysis requires the anomaly to have the form:

$$\delta_\sigma W = \int d^d x \sqrt{g}\, \sigma\, J \Delta_c J$$

where Δ_c is an operator of the form :

$$\Delta_c = \Box^n + \text{curvature terms}$$

such that

$$\Delta_c(e^{2\sigma} g) = e^{-(\frac{d}{2}+n)\sigma} \Delta_c(g) e^{+(\frac{d}{2}-n)\sigma}$$

The Gover-Hirachi theorem states that the operator exists **iff**

$1 \leq n \leq \frac{d}{2}$ for even d.

For example the operator appearing in the moduli anomalies ($n = 2$) in $d = 4$ is :

$$\Delta_c = \Box^2 + 2\nabla_\mu \Big(R^{\mu\nu} - \tfrac{1}{3} g^{\mu\nu} R \Big) \nabla \nu$$

the Fradkin-Tseytlin-Riegert operator.

However for $n > \frac{d}{2}$ i.e. **irrelevant** operators the anomaly in this form does not exist. The "physical reason " for the nonexistence can be seen in the OPE of two scalar operators:

$$\mathcal{O}(x)\mathcal{O}(0) \sim \frac{N}{x^{2\Delta}} + \ldots + \frac{a}{c_T} \frac{1}{x^{2\Delta+2-d}} T_{\alpha\beta}(0) x^\alpha x^\beta + \frac{a}{2c_T} \frac{1}{x^{2\Delta+2-d}} \partial_\gamma T_{\alpha\beta}(0) x^\alpha x^\beta x^\gamma$$

$$+ \frac{a}{8(d+3)c_T} \frac{1}{x^{2\Delta+2-d}} \Big((d+4) \partial_\gamma \partial_\delta T_{\alpha\beta}(0) x^\gamma x^\delta x^\alpha x^\beta - \Box T_{\alpha\beta}(0) x^2 x^\alpha x^\beta \Big) + \ldots$$

$$+ \frac{1}{x^{2\Delta - \tilde{\Delta} + n}} \tilde{\mathcal{O}}_{\mu_1 \ldots \mu_n} x^{\mu_1} \cdots x^{\mu_n}(0) + \ldots$$

The coefficient functions in front of the energy momentum tensors are logarithmically singular for irrelevant operators and the correlator which contributes to the anomaly will have a double logarithm. This is incompatible with the locality of the anomaly.

The singularity in the OPE signals a beta-function for the coupling of the energy momentum tensor i.e. for the metric. One should use the Osborn Equation even for two irrelevant sources since the metric has a beta-function i.e. the Weyl transformation operator is modified to:

$$\delta_\sigma W = \int d^d x \Big\{ \beta^g_{\mu\nu}(\bar{J}, g_{\mu\nu}, \sigma) \frac{\delta}{\delta g_{\mu\nu}(x)} + \sum_i \beta_i\big(\{J_j\}, g_{\mu\nu}\,\sigma\big) \frac{\delta}{\delta J_i(x)} \Big\} W$$

where $\beta^g_{\mu\nu}$ is the metric beta-function.

3. Metric beta-functions. Deformed Anomalies

Consider as an example a dimension five operator in $d = 4$. From the OPE the leading term in the beta function is :

$$\delta_\sigma g_{\mu\nu} = 2\,\sigma\, g_{\mu\nu} + \frac{a\,\pi^2}{24\,c_T}\sigma\,\rho\,\partial_\mu\partial_\nu\rho + \dots$$

Solving the first cohomology problem gives a solution for the complete beta-function:

$$\sigma\left(R_{\mu\nu}\,\rho^2 + (d-2)\rho\,\nabla_\mu\nabla_\nu\rho - (d-1)g_{\mu\nu}(\nabla\rho)^2 + g_{\mu\nu}\,\rho\,\Box\rho\right) \equiv \sigma\,\rho^2\,\hat{R}_{\mu\nu}$$

Here $\hat{R}$ is the curvature computed with the Weyl-invariant metric $\hat{g}_{\mu\nu} = \frac{1}{\rho^2} g_{\mu\nu}$.

Having the beta function one can solve the second cohomology problem for the determination of the anomaly:

$$
\begin{aligned}
\mathcal{A} = {} & c\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \alpha\, c \Big\{ \Box\rho\, \Box^2\rho - \tfrac{13}{8} R\, R^{\mu\nu} R_{\mu\nu}\, \rho^2 + \tfrac{53}{162} R^3\, \rho^2 + \tfrac{4}{3} R^{\mu\nu} R^{\rho\sigma} R_{\mu\rho\nu\sigma}\, \rho^2 \\
& - \tfrac{1}{8} R\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \rho^2 + \tfrac{43}{72} R_{\mu\nu\rho\sigma} R^{\mu\nu\alpha\beta} R_{\alpha\beta}{}^{\rho\sigma}\, \rho^2 - \tfrac{35}{72} R^2\, \rho\, \Box\rho + \tfrac{25}{24} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \rho\, \Box\rho \\
& - \tfrac{1}{36} \nabla^\mu R\, \nabla_\mu R\, \rho^2 + \tfrac{167}{12} R^{\mu\nu} R_{\mu\nu}\, \nabla^\alpha\rho\, \nabla_\alpha\rho - \tfrac{101}{24} R^2\, \nabla^\alpha\rho\, \nabla_\alpha\rho \\
& - \tfrac{79}{24} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \nabla^\alpha\rho\, \nabla_\alpha\rho - \tfrac{1}{3} R\, \Box\nabla^\mu\rho\, \nabla_\mu\rho - \tfrac{10}{9} R^{\mu\nu} \nabla_\mu\nabla_\nu R\, \rho^2 + \tfrac{7}{9} R^{\mu\nu} R\, \rho\, \nabla_\mu\nabla_\nu\rho \\
& + \tfrac{1}{36} \Box R\, \rho\, \Box\rho - \tfrac{16}{9} R\, (\Box\rho)^2 + \nabla^\mu R\, \nabla_\mu\rho\, \Box\rho + \tfrac{1}{6} R\, \rho\, \Box^2\rho - 4\, R^{\mu\nu}\, \nabla_\mu\rho\, \Box\nabla_\nu\rho \\
& - \tfrac{37}{18} R_{\mu\nu} \nabla^\mu R\, \rho\, \nabla_\nu\rho - 22\, R_\mu{}^\alpha R_{\nu\alpha}\, \nabla^\mu\rho\, \nabla^\nu\rho + \tfrac{116}{9} R^{\mu\nu}\, R\, \nabla^\mu\rho\, \nabla^\nu\rho \\
& - 13\, R^{\alpha\beta} R_{\mu\alpha\nu\beta}\, \nabla^\mu\rho\, \nabla^\nu\rho - \tfrac{5}{18} \nabla^\mu\nabla^\nu R\, \rho\, \nabla_\mu\nabla_\nu\rho - \tfrac{5}{9} R\, \nabla_\mu\nabla_\nu\rho\, \nabla^\mu\nabla^\nu\rho \\
& - 5\, R^{\beta\gamma}\, \nabla_\gamma R_{\alpha\beta}\, \rho\, \nabla^\alpha\rho - \tfrac{8}{3} R_\alpha{}^\gamma R^{\alpha\beta}\, \rho\, \nabla_\beta\nabla_\gamma\rho + \tfrac{10}{3} R^{\beta\gamma}\, \nabla^\alpha\rho\, \nabla_\gamma\nabla_\beta\nabla_\alpha\rho \\
& + \tfrac{5}{6} \Box R^{\mu\nu}\, \rho\, \nabla_\mu\nabla_\nu\rho + \tfrac{22}{3} R^{\mu\nu}\, \nabla_\mu\nabla_\nu\rho\, \Box\rho - \tfrac{5}{3} \nabla^\mu R^{\alpha\beta}\, \nabla_\mu R_{\alpha\beta}\, \rho^2 \Big\} + \mathcal{O}(\rho^4)
\end{aligned}
$$

The anomaly contains the universal coefficients c, α corresponding to the normalizations of the two energy-momentum tensors and two irrelevant operators correlators, respectively. In this sense the anomaly can be considered also as a deformation of the usual "c" conformal anomaly.

In a similar fashion there is a deformation of the "a" conformal anomaly:

$$
\begin{aligned}
\mathcal{A} = a\,E_4 + \alpha\, a \Big\{ & \tfrac{28}{135} R^3\,\rho^2 - \tfrac{6}{5}\, R^{\mu\nu}\, R^{\rho\sigma}\, R_{\mu\rho\nu\sigma}\,\rho^2 - \tfrac{7}{10}\, R\, R^{\mu\nu\rho\sigma}\, R_{\mu\nu\rho\sigma}\,\rho^2 \\
& + \tfrac{14}{15}\, R_{\mu\nu\rho\sigma} R^{\rho\sigma\alpha\beta} R_{\alpha\beta}{}^{\mu\nu}\,\rho^2 - \tfrac{1}{15} R\,\Box R\,\rho^2 + \tfrac{2}{3} R^2\,\rho\,\Box\rho + \tfrac{58}{5} R^{\mu\nu} R_{\mu\nu} (\nabla\rho)^2 \\
& - \tfrac{64}{15}\, R^2\,(\nabla\rho)^2 + \tfrac{1}{5} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}\,(\nabla\rho)^2 - 12\, R_{\mu\rho} R_{\nu}{}^{\rho}\, \nabla^\mu\rho\,\nabla^\nu\rho + 8\, R\, R_{\mu\nu}\,\nabla^\mu\rho\,\nabla^\nu\rho \\
& - 8\, R^{\rho\sigma} R_{\mu\rho\nu\sigma}\,\nabla^\mu\rho\,\nabla^\nu\rho + \tfrac{2}{5}\, R^{\mu\nu}\,\Box R_{\mu\nu}\,\rho^2 - 2\, R^{\mu\nu} R_{\mu\nu}\,\rho\,\Box\rho + \tfrac{7}{5}\,\nabla^\mu R^{\nu\rho}\,\nabla_\nu R_{\mu\rho}\,\rho^2 \\
& - \tfrac{7}{10}\,\nabla^\rho R^{\mu\nu}\,\nabla_\rho R_{\mu\nu}\,\rho^2 - \tfrac{4}{5}\nabla^\sigma\nabla^\rho R^{\mu\nu}\, R_{\mu\rho\nu\sigma}\,\rho^2 - \tfrac{7}{20}\,\nabla^\alpha R^{\mu\nu\rho\sigma}\,\nabla_\alpha R_{\mu\nu\rho\sigma}\,\rho^2 \Big\}
\end{aligned}
$$

This deformed anomaly does not introduce any new universal parameter.

Irrelevant operators to a certain order induce beta-functions for other operators: relevant,marginal and irrelevant up to a maximal dimension.

In $d = 4$ these beta-functions do not influence the metric beta-functions. The Weyl anomalies of the additional operators are deformed. The deformations can be related to the coupling of the initial irrelevant operators to energy-momentum tensors through the conformal blocks of the additional operators considered.

In $d = 2$ there is a subtle interplay between the induced beta-functions and the metric beta-function due to the presence of the $T\bar{T}$ operator.

Conclusions

-The presence of irrelevant operators requires the introduction of a metric beta-function in the Osborn Equation

-The Weyl anomalies are deformed and include the universal information contained in the beta-function

-Deforming the CFT by irrelevant operators to a certain order in PT produces a nontrivial "running" of the metric when the theory is formulated on a compact manifold

-Integrating the anomalies one can possibly relate the operator mixings and the partition function on a compact manifold (e.g. for the chiral ring of a SCFT)

-It would be interesting to understand the constraints implied by Weyl anomalies on the structure of the CFT when block decompositions related by crossing are used