Vortices, ergodicity and chaos from a holographic perspective

Marios Petropoulos

CPHT – Ecole Polytechnique – CNRS

Recreational thoughts on the occasion of Costas Kounnas's 60th birthday

Nicosia – September 2012

(inspiration: M. Caldarelli, R. Leigh, A. Petkou, V. Pozzoli and K. Siampos)

Highlights

Motivations

Generalities on holographic fluids

Fluids with vorticity

Magnetic flows

Outlook

Why vortices?

Developments in ultra-cold-atom physics: new twists in the physics of near-perfect neutral fluids fast rotating in normal or superfluid phase

Below BEC-transition: rotation ($\sim 100\,\mathrm{Hz}$) creates networks of vortices

He^4	$\bar{a} \sim 10^{-5}\,\mu\mathrm{m}$, $T_c \approx 2.17\,\mathrm{K}$, $a_v \sim 1\,\mathrm{mm} \Rightarrow$ a few highly populated vortices
(1995) BEC	$\bar{a} \sim 0.25\,\mu\mathrm{m}$, $T_c \sim 100\,\mathrm{nK}$, $a_v \sim 2\,\mu\mathrm{m} \Rightarrow$ 100 to 200 vortices

$\rightarrow$ *new challenges in strong-coupling regimes*

Rotation for neutral particles $\cong$ magnetic field for charged particles

Dilute rotating Bose gases in harmonic traps

- Landau levels with small filling factor (particles/vortices ≈ 1)
- Fractional-quantum-Hall effect
- Topological (anyonic) quasi-particle spectra

[e.g. Cooper *et al.* '10, Chu *et al.* '10, Dalibard *et al.* '11]

Foreseeable progress in the measurement of transport coefficients calls for a better understanding of the strong-coupling dynamics of vortices

Aim

Use AdS/CFT for

- *The description of rotating fluids as the hydrodynamic approximation of a boundary theory*
- *As a spin-off: the ergodicity breakdown in magnetic flows*

Highlights

Holographic duality

Applied beyond the original framework – maximal susy YM in $D = 4$ – usually in the classical gravity approximation without backreaction

- Bulk with $\Lambda = -3k^2$: asymptotically AdS $d = D + 1$-dim $\mathcal{M}$
- Conformal boundary at $r \to \infty$

Holography: determination of $\langle \mathcal{O} \rangle_{\text{bry. F.T.}}$ as a response to a boundary source perturbation $\delta\phi_{(0)}$

Where is the fluid and where are its data?

Via holography: boundary field theory at finite T and μ

Fluid $\equiv$ hydrodynamic approximation of the boundary theory

- stationary $\longrightarrow$ local thermodynamic equilibrium – no dissipation
- disturbed $\longrightarrow$ dissipative response – alternative to kinetic theory

Fluid described in terms of $\mathbf{u}, \varepsilon, p, T$ in $T_{\mu\nu}$ s.t. $\nabla_\mu T^{\mu\nu} = 0$ in a background $g_{(0)\mu\nu}$ – data read off from the large-r expansion of θ^μ

[Palatini formulation and $\mathbf{3+1}$ split – Leigh, Petkou '07, Mansi, Petkou, Tagliabue '08]

θ^a: bulk orthonormal coframe $(\eta : + - ++)$

$$ds^2 = \eta_{ab}\theta^a\theta^b$$

with a gauge choice s.t. $\theta^r = \frac{dr}{kr}, \quad \theta^\mu = \theta^\mu{}_\nu dx^\nu, \quad \mu = 0, 1, 2$

Holography: Hamiltonian evolution from data on the boundary – captured in Fefferman–Graham expansion for large r [Fefferman, Graham '85]

$$\theta^\mu(r,x) = kr\,E^\mu(x) + \frac{1}{kr}F^\mu_{[2]}(x) + \frac{1}{k^2r^2}F^\mu(x) + \cdots$$

Independent $2+1$ boundary data: vector-valued 1-forms E^μ and F^μ

- E^μ: boundary orthonormal coframe – allows to determine
$$\mathrm{d}s^2_{\mathrm{bry.}} = g_{(0)\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \eta_{\mu\nu}E^\mu E^\nu$$
- F^μ: stress current one-form – allows to construct the boundary stress tensor ($\kappa = {}^{3k}/_{8\pi G}$)
$$T = \kappa F^\mu e_\mu = T^\mu{}_\nu E^\nu \otimes e_\mu$$

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Basic ingredients

Reminder: $u \to \nabla_\mu u_\nu \to \{a_\mu, \sigma_{\mu\nu}, \Theta, \omega_{\mu\nu}\}$

$$\omega = \frac{1}{2}\left(\mathrm{d}u + u \wedge a\right)$$

- Global rigid rotation: AdS Kerr (M, a) – boundary fluid on a squashed two-sphere in cyclonic motion *i.e.* with

$$\omega \sim a \cos\vartheta \sin\vartheta \, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi \sim a \cos\vartheta E^\vartheta \wedge E^\varphi$$

- Local rotation: nut-like charge – fluid in homogeneous *monopolar* rotation on S^2 or H_2 with *constant* vorticity

AdS Taub–NUT: the nut charge

The bulk data *[Taub '51, Newman, Tamburino, Unti '63]*

$$\begin{aligned} \mathrm{d}s^2 &= (\theta^r)^2 - (\theta^t)^2 + (\theta^\vartheta)^2 + (\theta^\varphi)^2 \\ &= \frac{\mathrm{d}\tilde{r}^2}{V(\tilde{r})} - V(\tilde{r})\left[\mathrm{d}t - 2n\cos\vartheta\,\mathrm{d}\varphi\right]^2 + \rho^2\left[\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi\right]^2 \end{aligned}$$

$V(\tilde{r}) = {}^{\Delta}/_{\rho^2}$ with

$$\begin{aligned} \Delta &= \left(\tilde{r}^2 - n^2\right)\left(1 + k^2\left(\tilde{r}^2 + 3n^2\right)\right) + 4k^2n^2\tilde{r}^2 - 2M\tilde{r} \\ \rho^2 &= \tilde{r}^2 + n^2 \end{aligned}$$

Horizon, no angular velocity a *but nut charge* n *– one of the most peculiar solutions to Einstein's Eqs.* *[Misner '63]*

Taub–NUT: rich geometry – foliation over squashed 3-spheres with $\mathbb{R} \times SU(2)$ isometry (homogeneous and axisymmetric)

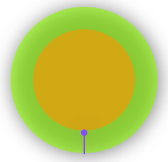

Figure: horizon (bolt), nut and Misner string (coordinate singularity)

Following FG → boundary metric and stress tensor

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{bry.}} &= \eta_{\mu\nu} E^\mu E^\nu = g_{(0)\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -\left(\mathrm{d}t - 2n(\cos\vartheta - 1)\mathrm{d}\varphi\right)^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \end{aligned}$$

$$T = T_{\mu\nu} E^\mu E^\nu = \frac{\kappa M k}{3}\left(2(E^t)^2 + (E^\vartheta)^2 + (E^\varphi)^2\right)$$

Fluid interpretation: perfect-like stress tensor

- conformal with $\varepsilon = 2p = {}^{2\kappa Mk}/_{3}$
- velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial ($\nabla_{\partial_t}\partial_t = 0$)

no expansion, no shear but *vorticity*

The vorticity on the boundary of AdS Taub–NUT ($u = -\mathrm{d}t + b$)

$$b = -2n(1-\cos\vartheta)\mathrm{d}\varphi = -2n\frac{1-\cos\vartheta}{\sin\vartheta}E^{\varphi}$$

$$\omega = \frac{1}{2}\mathrm{d}b = -n\sin\vartheta\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi = -nk^2E^{\vartheta}\wedge E^{\varphi}$$

Dirac-monopole-like *vortex* (“hedgehog” or homogeneous)

Taub–NUT is well designed for describing “monopolar” vortices (Kerr produces a dipole without nut charge: $\int\omega = 0$ – solid rotation)

Reminder

Rotation in flat space (spherical coordinates)

Data: $\vec{v}$ $\quad \vec{\omega} = {}^1\!/\!{}_2 \vec{\nabla} \times \vec{v}$

- Solid rotation ($\ell = 2$):
  - $\vec{v} = \Omega \partial_\varphi$ and $\|\vec{v}\| = \Omega r \sin\vartheta$ (regular)
  - $\vec{\omega} = \Omega \cos\vartheta \partial_r - \frac{\Omega \sin\vartheta}{r} \partial_\vartheta = \Omega \partial_z$ (uniform)
- Ordinary vortex ($\ell = 0$):
  - $\vec{v} = \frac{\beta}{r^2 \sin^2\vartheta} \partial_\varphi$ and $\|\vec{v}\| = \frac{\beta}{r \sin\vartheta}$ (singular at $\vartheta = 0, \pi$)
  - $\vec{\omega} = 0$ (irrotational) – up to a δ-function contribution
- Dirac-monopole vortex ($\ell = 1$):
  - $\vec{v} = \alpha \frac{1 - \cos\vartheta}{r^2 \sin^2\vartheta} \partial_\varphi$ and $\|\vec{v}\| = \alpha \frac{1 - \cos\vartheta}{r \sin\vartheta}$ (singular at $\vartheta = \pi$)
  - $\vec{\omega} = \frac{\alpha}{2r^2} \partial_r$ (hedgehog)

Spherical vs. hyperbolic Taub–NUT

Similar bulk solution exists with hyperbolic horizon, nut charge sent away and no Misner string – similar holographic fluid interpretation

- Boundary metric: squashed AdS_3 ($\mathbb{R} \times SL(2, \mathbb{R})$ isometry)

$$\mathrm{d}s^2_{\text{bry.}} = -\left(\mathrm{d}t - 2n(\cosh\sigma - 1)\mathrm{d}\varphi\right)^2 + \frac{1}{k^2}\left(\mathrm{d}\sigma^2 + \sinh^2\sigma\mathrm{d}\varphi^2\right)$$

- Vorticity: homogeneous ("hedgehog" on H_2 – Poincaré plane)

$$\omega = n\sinh\sigma\,\mathrm{d}\sigma \wedge \mathrm{d}\varphi = nk^2 E^\sigma \wedge E^\varphi$$

Paradigm for magnetic flows

Highlights

Papapetrou–Randers time fibrations

Stationary bulk solutions lead to boundaries of the generic form

$$\mathrm{d}s^2_{\text{bry.}} = -\left(\mathrm{d}t - b_k \mathrm{d}x^k\right)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j$$

with inertial, shear- and expansion-less fluid s.t. $\omega = {}^1\!/_2\, \mathrm{d}b$

- Not always globally hyperbolic: $\exists$ regions on the base where constant-t surfaces are *not* space-like – happens wherever

$$a^{ij} b_i b_j > 1$$

- Mapped to *magnetic flows*

  Geodesic flows on PR $\leftrightarrow$ magnetic flows on $\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i \mathrm{d}x^j$

Geodesics and magnetic mapping

PR is stationary: ∂_t is Killing and "energy" is conserved

- In proper time τ

$$\mathcal{E} = \dot{t} - b_i \dot{x}^i$$
$$\mathcal{E}^2 - \kappa = a_{ij}\dot{x}^i\dot{x}^j$$

  $\kappa = 0, 1$ for light- or time-like geodesics

- In time $\zeta = \mathcal{E}\tau$: dynamics captured by the Newtonian system

$$\tilde{L} = \frac{1}{2} a_{ij} x'^i x'^j + b_k x'^k$$
$$\tilde{\mathcal{E}} = \frac{1}{2} a_{ij} x'^i x'^j = \frac{1}{2}\left(1 - {}^{\kappa}\!/\!_{\mathcal{E}^2}\right)$$

Charged particle on $\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j$ in a magnetic field $F = \mathrm{d}b$ ($m = e = 1$) with the following mapping of "Newtonian" energy

$$\tilde{\mathcal{E}} = {}^1\!/\!_2 \leftrightarrow \textit{light-like} \qquad 0 \leq \tilde{\mathcal{E}} < {}^1\!/\!_2 \leftrightarrow \textit{time-like}$$

Application: squashed AdS_3

Geodesic congruence on squashed AdS_3

$\updownarrow$

Magnetic flow on Poincaré plane with uniform magnetic field

Playground for ergodicity analysis – reminder

- Dynamical system: $\{\mathcal{M}, \Phi_\zeta, \mu\}$
- Ergodic iff for any observable f

$$\int_{\mathcal{M}} f \, \mathrm{d}\mu = \lim_{T\to\infty} \frac{1}{T} \int_0^T f\left(\Phi_\zeta(x)\right) \mathrm{d}\zeta \quad \text{almost } \forall x \in \mathcal{M}$$

In practice: flow sufficiently unstable to guarantee that any open subset of $\mathcal{M}$ be mapped onto the entire $\mathcal{M}$

Geodesic flows on negative-curvature spaces are ergodic [Hadamard; Hopf; '30s]
(positive-entropy, mixing, infinite-multiplicity Lebesgue spectrum)

Poincaré plane is the simplest

- 2-dim
- constant-curvature
- $SL(2,\mathbb{R})$-invariant

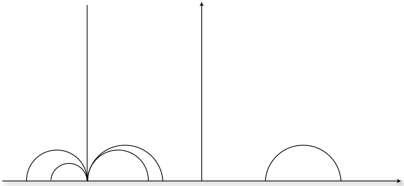

Figure: Geodesic flow on Poincaré plane

Magnetic flow on Poincaré plane: ergodic for small magnetic field w.r.t. the energy [Hedlund, 1936]

With a *uniform* magnetic field $2nk^2$ – still $SL(2,\mathbb{R})$-invariant

- ergodic if $2nk < \sqrt{2\tilde{\mathcal{E}}}$
- non-ergodic otherwise (periodic)

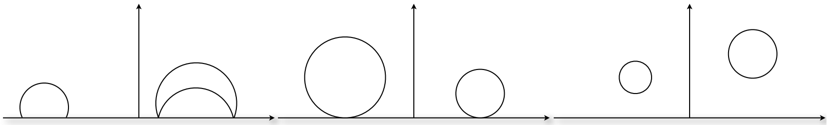

Figure: Magnetic flow on Poincaré plane: $2nk \lesseqgtr \sqrt{2\tilde{\mathcal{E}}}$

Competition magnetic field / negative curvature: absent in flat space or on S^2

Squashed-AdS_3 perspective and holography

For light- or time-like geodesics on squashed AdS_3 $0 \leq \tilde{\mathcal{E}} < 1/2$

Those correspond to ergodic magnetic flow on the Poincaré plane iff

$$2kn < \sqrt{2\tilde{\mathcal{E}}} \leq 1$$

- $2nk < 1$ is precisely the requirement for global hyperbolicity in squashed AdS_3
- $2nk = 1$ seems to be the endpoint of a transition/bifurcation line in the thermodynamics of the holographic dual – the hyperbolic Taub–NUT [under investigation]

Highlights

A web of relations

Around a class of 4-dim bulk solutions with nut charge – usually considered as a curiosity

- *Appearance of vortices*
- *Ergodicity and its breakdown – valid also for the Schrödinger spectrum*

Yet to be unravelled

- *Transport coefficients s.t. Hall viscosity*
- *Thermodynamic properties*
- *Analogue-gravity paradigm*

Workshop on holographic applications, out-of-equilibrium phenomena, gravity & analogue gravity

Speakers

Iacopo Carusotto (Trento)
Mihalis Dafermos (Cambridge)
Gianguido Dall'Agata (Padua)
Manuel Floratos (Athens)
Dmitry Fursaev (Dubna)
Jerome Gauntlett (Imperial)
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Highlights

Holography

Applied beyond the original framework – maximal susy YM in $D = 4$ – usually in the classical gravity approximation without backreaction

- Bulk: "asymptotically AdS" d-dim $\mathcal{M}$ $(d = D + 1)$

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{k^2 r^2} + k^2 r^2 H(kr) \left(-\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2\right)$$

- Boundary at $r \to \infty$: $\mathrm{d}s^2 \approx \frac{\mathrm{d}r^2}{k^2 r^2} + k^2 r^2 g_{(0)\mu\nu}(x) \mathrm{d}x^\mu \mathrm{d}x^\nu$
- Dynamical field ϕ with action $I\left[\phi\right]$ and boundary value $\phi_{(0)}(x)$

The basic relation

$$Z_{\mathsf{bulk}}[\phi] = \langle 1 \rangle_{\mathsf{bry.\ F.T.}}$$

gives access to the data of the boundary theory

$$\left\langle \exp i \int_{\partial\mathcal{M}} \mathrm{d}^D x \sqrt{-g_{(0)}}\, \delta\phi_{(0)}\, \mathcal{O} \right\rangle_{\mathsf{bry.\ F.T.}} = Z_{\mathsf{bulk}}[\phi + \delta\phi_{(0)}]$$

- $\phi_{(0)} \leftrightarrow \mathcal{O}$: conjugate variables
- $\delta\phi_{(0)}$: boundary perturbation $\longrightarrow$ source
- $\mathcal{O}$: observable functional of $\phi_{(0)}$ $\longrightarrow$ response

Semi-classically around a classical solution $\phi_\star$

$$Z_{\text{bulk}}[\phi] = \exp - I_{\text{E}}[\phi_\star]$$
$$\langle \mathcal{O} \rangle = \left. \frac{\delta I}{\delta \phi_{(0)}} \right|_{\phi_\star}$$

Hamiltonian interpretation of $\langle \mathcal{O} \rangle$

- $\pi = \frac{\partial \mathcal{L}}{\partial \partial_r \phi} \Rightarrow I = \int \mathrm{d}r \int \mathrm{d}^D x \left[\pi \partial_r \phi - \mathcal{H}(\pi, \phi, \partial_\mu \phi) \right]$
- on-shell variation

$$\delta I|_{\phi_\star} = \int_{\partial \mathcal{M}} \mathrm{d}^D x \, \pi_{(0)} \, \delta \phi_{(0)} \Rightarrow \langle \mathcal{O} \rangle = \pi_{(0)}$$

What is holography? How do we get $\pi_{(0)} = \pi_{(0)}\left[\phi_{(0)}\right]$?

$$\partial\mathcal{M} = \begin{cases} \text{boundary } r \to \infty \\ \text{horizon } r_{\mathrm{H}} \end{cases}$$

- $\phi_{(0)}(x)$ and $\pi_{(0)}(x)$ are *independent* data set at large r

$$\phi(r) = r^{\Delta-d}\phi_{(0)}(x) + \frac{r^{-\Delta}}{k(2\Delta - D)}\pi_{(0)}(x) + \cdots$$

  (non-normalizable and normalizable modes)
- become related if a *regularity condition* is imposed at r_{H}

$$\langle\mathcal{O}\rangle = \pi_{(0)}\left[\phi_{(0)}\right]$$

In summary

Holography: determination of $\langle \mathcal{O} \rangle_{\text{bry. F.T.}}$ – unknown microscopic theory – as a response to a boundary source perturbation $\delta\phi_{(0)}$

- Dynamical field ϕ with action $I\left[\phi\right]$ and boundary value $\phi_{(0)}(x)$
- Momentum $\pi(r, x)$ with boundary value $\pi_{(0)}(x)$
- On-shell variation

$$\delta I|_{\phi_\star} = \int_{\partial\mathcal{M}} \mathrm{d}^D x \, \pi_{(0)} \, \delta\phi_{(0)}$$

- Holography: regularity on $r_{\mathrm{H}} \Rightarrow \pi_{(0)} = \pi_{(0)}\left[\phi_{(0)}\right] \longrightarrow$ semiclassically

$$\langle \mathcal{O} \rangle = \left.\frac{\delta I}{\delta\phi_{(0)}}\right|_{\phi_\star} = \pi_{(0)}\left[\phi_{(0)}\right]$$

Examples

Electromagnetic field in $d = 4$, $D = 3$

- Field $A_r, A_\mu \longrightarrow A_{(o)\mu}$: boundary electromagnetic field – source
- Momentum $\mathcal{E}_\mu \longrightarrow \mathcal{E}_{(0)\mu}$: $\langle \varrho \rangle$, $\langle j_i \rangle$ – response
- Bulk gauge invariance $\longrightarrow$ continuity equation

Gravitation in $d = D + 1$

- Field $g_{rr}, g_{\mu\nu} \longrightarrow g_{(o)\mu\nu}$: boundary metric – source
- Momentum $T_{\mu\nu} \longrightarrow T_{(o)\mu\nu}$: $\langle T_{(o)\mu\nu} \rangle$ – response
- Bulk diffeomeorphism invariance $\longrightarrow$ conservation equation

Gravity in $d = 4$

Palatini formulation and $3+1$ *split* [Leigh, Petkou '07, Mansi, Petkou, Tagliabue '08]

$$I_{\mathrm{EH}} = -\frac{1}{32\pi G}\int_{\mathcal{M}} \epsilon_{abcd}\left(\mathcal{R}^{ab} - \frac{\Lambda}{3}\theta^a \wedge \theta^b\right) \wedge \theta^c \wedge \theta^d$$

θ^a an orthonormal frame $\mathrm{d}s^2 = \eta_{ab}\theta^a\theta^b$ ($\eta : + - + +$)

- Vierbein: $\theta^r = N\frac{\mathrm{d}r}{kr} \quad \theta^\mu = N^\mu \mathrm{d}r + \tilde{\theta}^\mu \quad \mu = 0, 1, 2$

$$\mathrm{d}s^2 = N^2 \frac{\mathrm{d}r^2}{k^2 r^2} + \eta_{\mu\nu}\left(N^\mu \mathrm{d}r + \tilde{\theta}^\mu\right)\left(N^\nu \mathrm{d}r + \tilde{\theta}^\nu\right)$$

- Connection: $\omega^{r\mu} = q^{r\mu}\mathrm{d}r + \mathcal{K}^\mu \quad \omega^{\mu\nu} = -\epsilon^{\mu\nu\rho}\left(Q_\rho \frac{\mathrm{d}r}{kr} + \mathcal{B}_\rho\right)$

(note: $\Lambda = -3k^2$)

Aim: Hamiltonian evolution from data on the boundary $r \to \infty$
Question: what are the field and momentum variables?

- Gauge choice: $N = 1$ and $N^{\mu} = q^{r\mu} = Q_{\rho} = 0$

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{k^2 r^2} + \eta_{\mu\nu}\tilde{\theta}^{\mu}\tilde{\theta}^{\nu}$$

- Fields and momenta: $\tilde{\theta}^{\mu}, \mathcal{K}^{\mu}, \mathcal{B}_{\rho}$ one-forms

What are the independent boundary data? Answer in asymptotically AdS: Fefferman–Graham expansion for large r [Fefferman, Graham '85]

$$\begin{aligned}
\tilde{\theta}^\mu(r,x) &= kr\, E^\mu(x) + \tfrac{1}{kr} F^\mu_{[2]}(x) + \tfrac{1}{k^2 r^2} F^\mu(x) + \cdots \\
\mathcal{K}^\mu(r,x) &= -k^2 r\, E^\mu(x) + \tfrac{1}{r} F^\mu_{[2]}(x) + \tfrac{2}{kr^2} F^\mu(x) + \cdots \\
\mathcal{B}^\mu(r,x) &= B^\mu(x) + \tfrac{1}{k^2 r^2} B^\mu_{[2]}(x) + \cdots
\end{aligned}$$

Independent $2+1$ boundary data: E^μ and F^μ

Upon canonical transformations (i.e. boundary terms or holographic renormalization)

$$\delta I_{\mathrm{EH}}\big|_{\mathrm{on-shell}} = \int_{\partial\mathcal{M}} T^\mu \wedge \delta\Sigma_\mu$$

- $\Sigma_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho} E^\nu \wedge E^\rho$: field – source
- $T^\mu = \kappa F^\mu$: momentum – response

Application: Schwartzschild AdS

The bulk data

$$\mathrm{d}s^2 = \frac{\mathrm{d}\tilde{r}^2}{V(\tilde{r})} - V(\tilde{r})\mathrm{d}t^2 + \tilde{r}^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)$$

- $V(r) = 1 + k^2\tilde{r}^2 - {}^{2M}/_{\tilde{r}}$
- $\theta^r = {}^{\mathrm{d}\tilde{r}}/_{\sqrt{V(\tilde{r})}} = {}^{\mathrm{d}r}/_{kr}$

The Fefferman–Graham expansion

$$\begin{aligned}
\theta^t &= \sqrt{V(\tilde{r})}\mathrm{d}t = \left(kr + \tfrac{1}{4kr} - \tfrac{2M}{3kr^2} + \mathrm{O}\left(\tfrac{1}{r^3}\right)\right)\mathrm{d}t\\
\theta^\vartheta &= \tilde{r}\,\mathrm{d}\vartheta = \left(r - \tfrac{1}{4k^2r} + \tfrac{M}{3k^2r^2} + \mathrm{O}\left(\tfrac{1}{r^3}\right)\right)\mathrm{d}\vartheta\\
\theta^\varphi &= \tilde{r}\sin\vartheta\,\mathrm{d}\varphi = \left(r - \tfrac{1}{4k^2r} + \tfrac{M}{3k^2r^2} + \mathrm{O}\left(\tfrac{1}{r^3}\right)\right)\sin\vartheta\,\mathrm{d}\varphi
\end{aligned}$$

The boundary data

- coframe: $E^t = \mathrm{d}t \quad E^\vartheta = \frac{\mathrm{d}\vartheta}{k} \quad E^\varphi = \frac{\sin\vartheta\,\mathrm{d}\varphi}{k}$
- stress current: $F^t = -\frac{2Mk}{3}\mathrm{d}t \quad F^\vartheta = \frac{M}{3}\mathrm{d}\vartheta \quad F^\varphi = \frac{M}{3}\sin\vartheta\,\mathrm{d}\varphi$

The boundary metric

$$\begin{aligned} \mathrm{d}s^2_{\text{bry.}} &= \eta_{\mu\nu}E^\mu E^\nu = g_{(0)\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -\mathrm{d}t^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right) \end{aligned}$$

- Einstein universe
- $e_t = \partial_t$
- $\nabla_{e_t} e_t = 0$: observers at rest are inertial

The boundary stress tensor $\kappa F^{\mu} e_{\mu}$

$$T = T_{\mu\nu} E^{\mu} E^{\nu} = \frac{\kappa M k}{3}\left(2(E^{t})^{2} + (E^{\vartheta})^{2} + (E^{\varphi})^{2}\right)$$

- traceless: conformal fluid with $\varepsilon = 2p = {}^{2\kappa Mk}/{}_{3}$
- velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial
- velocity one-form: $u = -E^{t} = -\mathrm{d}t$

Static fluid without expansion, shear or vorticity

Notes

The fluid may be perfect or not

$$T_{\text{visc}} = -\left(2\eta\sigma^{\mu\nu} + \zeta h^{\mu\nu}\Theta\right) e_\mu \otimes e_\nu$$

$T_{\text{visc}} = 0$ if the congruence is shear- and expansion-less

A shear- and expansion-less isolated fluid is geodesic if [Caldarelli et al. '08]

$$\nabla_{\mathbf{u}}\varepsilon = 0$$

$$\nabla p + u\nabla_{\mathbf{u}} p = 0$$

fulfilled here with ε, p csts.

Only $\delta g_{(\mathrm{o})\mu\nu}$ *give access to* η *and* ζ *via* $\langle \delta T_{(\mathrm{o})\mu\nu} \rangle$

More general examples

We can exhibit backgrounds with stationary boundaries and fluids

$$T = (\varepsilon + p)\mathbf{u} \otimes \mathbf{u} + p\eta^{\mu\nu} e_\mu \otimes e_\nu$$

- $\varepsilon = 2p$: conformal
- $\nabla_{\mathbf{u}}\mathbf{u} = 0$: inertial
- $\mathbf{u} = e_0$: at rest (comoving)

On vector-field congruences [Ehlers '61]

Vector field $\mathbf{u}$ *with* $u_\mu u^\mu = -1$ *and space–time variation* $\nabla_\mu u_\nu$

$$\nabla_\mu u_\nu = -u_\mu a_\nu + \sigma_{\mu\nu} + \frac{1}{D-1}\Theta h_{\mu\nu} + \omega_{\mu\nu}$$

- $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$: projector/metric on the orthogonal space
- $a_\mu = u^\nu \nabla_\nu u_\mu$: acceleration – transverse
- $\sigma_{\mu\nu}$: symmetric traceless part – shear
- $\Theta = \nabla_\mu u^\mu$: trace – expansion
- $\omega_{\mu\nu}$: antisymmetric part – vorticity

$$\omega = \frac{1}{2}\omega_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}(\mathrm{d}u + u \wedge a)$$

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AdS Kerr: the solid rotation

The bulk data

$$\begin{aligned}
\mathrm{d}s^2 &= (\theta^r)^2 - (\theta^t)^2 + (\theta^\vartheta)^2 + (\theta^\varphi)^2 \\
&= \frac{\mathrm{d}\tilde{r}^2}{V(\tilde{r},\vartheta)} - V(\tilde{r},\vartheta)\left[\mathrm{d}t - \frac{a}{\Xi}\sin^2\vartheta\,\mathrm{d}\varphi\right]^2 \\
&+ \frac{\rho^2}{\Delta_\vartheta}\mathrm{d}\vartheta^2 + \frac{\sin^2\vartheta\Delta_\vartheta}{\rho^2}\left[a\,\mathrm{d}t - \frac{r^2+a^2}{\Xi}\mathrm{d}\varphi\right]^2
\end{aligned}$$

$V(\tilde{r},\vartheta) = {}^{\Delta}/_{\rho^2}$ with

$$\begin{aligned}
\Delta &= \left(\tilde{r}^2 + a^2\right)\left(1 + k^2\tilde{r}^2\right) - 2M\tilde{r} \\
\rho^2 &= \tilde{r}^2 + a^2\cos^2\vartheta \\
\Delta_\vartheta &= 1 - k^2a^2\cos^2\vartheta \\
\Xi &= 1 - k^2a^2 > 0
\end{aligned}$$

The boundary metric – following FG expansion

$$\begin{aligned} \mathrm{d}s^2_{\text{bry.}} &= \eta_{\mu\nu} E^\mu E^\nu = g_{(0)\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -\left(\mathrm{d}t - \frac{a \sin^2\vartheta}{\Xi} \mathrm{d}\varphi\right)^2 + \frac{1}{k^2 \Delta_\vartheta}\left(\mathrm{d}\vartheta^2 + \left(\frac{\Delta_\vartheta \sin\vartheta}{\Xi}\right)^2 \mathrm{d}\varphi^2\right) \end{aligned}$$

- spatial section: squashed 2-sphere
- $\nabla_{\partial_t}\partial_t = 0$: observers at rest are *inertial*
- note: conformal to Einstein universe in a *rotating frame* (requires $(\vartheta, \varphi) \to (\vartheta', \varphi')$)

The boundary stress tensor $\kappa F^{\mu} e_{\mu}$ *[see also Caldarelli, Dias, Klemm '08]*

$$T = T_{\mu\nu} E^{\mu} E^{\nu} = \frac{\kappa M k}{3} \left(2(E^t)^2 + (E^{\vartheta})^2 + (E^{\varphi})^2 \right)$$

perfect-fluid-like ($T = (\varepsilon + p) u \otimes u + p \eta_{\mu\nu} E^{\mu} \otimes E^{\nu}$)

- traceless: conformal fluid with $\varepsilon = 2p = {}^{2\kappa Mk}/_{3} \propto T^2$
- velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial

Vorticity but no expansion or shear – the viscosity η, ζ is not felt

$$\omega = \frac{1}{2} \mathrm{d}u = \frac{1}{2} \mathrm{d}b = \frac{a \cos\vartheta \sin\vartheta}{\Xi} \mathrm{d}\vartheta \wedge \mathrm{d}\varphi = k^2 a \cos\vartheta E^{\vartheta} \wedge E^{\varphi}$$

Reminder: $u \to \nabla_{\mu} u_{\nu} \to \{a_{\mu}, \sigma_{\mu\nu}, \Theta, \omega_{\mu\nu}\}$

Highlights

AdS Taub–NUT: the nut charge

Reminder: the bulk data *[Taub '51, Newman, Tamburino, Unti '63]*

$$\mathrm{d}s^2 = \frac{\mathrm{d}\tilde{r}^2}{V(\tilde{r})} - V(\tilde{r})\left[\mathrm{d}t - 2n\cos\vartheta\,\mathrm{d}\varphi\right]^2 + \rho^2\left[\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi\right]^2$$

$V(\tilde{r}) = {}^{\Delta}/_{\rho^2}$ with

$$\begin{aligned}
\Delta &= \left(\tilde{r}^2 - n^2\right)\left(1 + k^2\left(\tilde{r}^2 + 3n^2\right)\right) + 4k^2n^2\tilde{r}^2 - 2M\tilde{r}\\
\rho^2 &= \tilde{r}^2 + n^2
\end{aligned}$$

The Fefferman–Graham expansion with r s.t. $\mathrm{d}r/_{kr} = \mathrm{d}\tilde{r}/\sqrt{V(\tilde{r})}$

- boundary coframe and frame

$$E^t = \mathrm{d}t - b \quad E^\vartheta = \tfrac{\mathrm{d}\vartheta}{k} \quad E^\varphi = \tfrac{\sin\vartheta\,\mathrm{d}\varphi}{k}$$
$$e_t = \partial_t \quad e_\vartheta = k\,\partial_\vartheta \quad e_\varphi = -\tfrac{2kn(1-\cos\vartheta)}{\sin\vartheta}\partial_t + \tfrac{k}{\sin\vartheta}\partial_\varphi$$

$$b = -2n(1-\cos\vartheta)\mathrm{d}\varphi$$

- boundary stress current

$$F^t = -\frac{2Mk}{3}E^t \quad F^\vartheta = \frac{Mk}{3}E^\vartheta \quad F^\varphi = \frac{Mk}{3}E^\varphi$$

For comparison: AdS Kerr

The Fefferman–Graham expansion of $\theta^t, \theta^\vartheta, \theta^\varphi$

- boundary orthonormal coframe and frame

$$E^t = \mathrm{d}t - b \quad E^\vartheta = \frac{\mathrm{d}\vartheta}{k\sqrt{\Delta_\vartheta}} \quad E^\varphi = \frac{\sqrt{\Delta_\vartheta}\sin\vartheta\,\mathrm{d}\varphi}{k\Xi}$$
$$e_t = \partial_t \quad e_\vartheta = k\sqrt{\Delta_\vartheta}\,\partial_\vartheta \quad e_\varphi = \frac{ka\sin\vartheta}{\sqrt{\Delta_\vartheta}}\partial_t + \frac{k\Xi}{\sin\vartheta\sqrt{\Delta_\vartheta}}\partial_\varphi$$

$$b = \frac{a\sin^2\vartheta}{\Xi}\mathrm{d}\varphi$$

- boundary stress current

$$F^t = -\frac{2Mk}{3}E^t \quad F^\vartheta = \frac{Mk}{3}E^\vartheta \quad F^\varphi = \frac{Mk}{3}E^\varphi$$

The boundary metric and stress tensor

$$\begin{aligned} \mathrm{d}s^2_{\text{bry.}} &= \eta_{\mu\nu} E^\mu E^\nu = g_{(0)\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -\left(\mathrm{d}t + 2n(1-\cos\vartheta)\mathrm{d}\varphi\right)^2 + \tfrac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \end{aligned}$$

$$T = T_{\mu\nu} E^\mu E^\nu = \frac{\kappa M k}{3}\left(2(E^t)^2 + (E^\vartheta)^2 + (E^\varphi)^2\right)$$

Fluid interpretation: perfect-like stress tensor

- conformal fluid with $\varepsilon = 2p = {}^{2\kappa Mk}/{}_{3}$
- velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial

Fluid without expansion and shear but *with vorticity*

$$\omega = \frac{1}{2}\mathrm{d}b = -n\sin\vartheta \mathrm{d}\vartheta \wedge \mathrm{d}\varphi = -k^2 n E^\vartheta \wedge E^\varphi$$

How does vorticity i.e. rotation get manifest?

Boundary geometries are stationary of Randers form [Randers '41]

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - b\right)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

and the fluid is at rest: $\mathbf{u} = \partial_t$

- $\nabla_{\partial_t}\partial_t = 0$: the fluid is inertial and carries vorticity $\omega = \frac{1}{2}\mathrm{d}b$
- $\nabla_{\partial_t}\partial_i = \omega_{ij}a^{jk}\left(\partial_k + b_k\partial_t\right)$: frame and fluid dragging

Other privileged frames exist where the observers experience differently the rotation of the fluid – e.g. Zermelo dual frame

AdS Taub–NUT: more on the boundary and CTCs

Homogenous boundary space–time: Lorentzian squashed 3-sphere

$$\begin{aligned} \mathrm{d}s^2_{\text{bry.}} &= \frac{1}{k^2}\left((\sigma^1)^2 + \left(\sigma^2\right)^2\right) - 4n^2\left(\sigma^3\right)^2 \\ &= -\left(\mathrm{d}t - 2n(\cos\vartheta - 1)\mathrm{d}\varphi\right)^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \end{aligned}$$

- Stationary foliation in 2-spheres with a *time fiber*
- Gödel-like space sourced by dust distribution [classification in Raychaudhuri *et al.* '80, Rebouças *et al.* '83]
- CTCs of angular opening $< 2\vartheta_0$ $(g_{\varphi\varphi}(\vartheta_0) = 0)$ – *no closed time-like geodesics*
- Special point: south pole of the 2-sphere – track of the Misner string – can be moved anywhere by homogeneity

Around the poles: Som–Raychaudhuri and cosmic spinning string

- North pole: Som–Raychaudhuri space – sourced by rigidly rotating charged dust [Som, Raychaudhuri '68]

$$\mathrm{d}s^2 = -\left(\mathrm{d}t + \Omega\varrho^2 \mathrm{d}\varphi\right)^2 + \varrho^2 \mathrm{d}\varphi^2 + \mathrm{d}\varrho^2$$

  $\Omega = k^2 n$ and $\varrho = {}^{\vartheta}/_{k}$

- South pole: spinning cosmic string [vortex in analogue gravity]

$$\mathrm{d}s^2 = -\left(\mathrm{d}t + A\mathrm{d}\varphi\right)^2 + \varrho^2 \mathrm{d}\varphi^2 + \mathrm{d}\varrho^2$$

  $A = 4n - \Omega\varrho^2$ and $\varrho = {}^{\pi-\vartheta}/_{k}$

Around the poles of Kerr: Som–Raychaudhuri with $\Omega = -k^2 a$

Kerr vs. Taub–NUT "rotation" [Dowker '74, Bonnor '75, Hunter '98]

- Kerr: rigid rotation with angular momentum and velocity
  - horizon at $r = r_+$: fixed locus of $\partial_t + \Omega_\mathrm{H}\partial_\varphi \to$ bolt
  - pair of nut–anti-nut at $r = r_+, \vartheta = 0, \pi$ (fixed points of ∂_φ) connected by a Misner string [Argurio, Dehouck '09]

  asymptotically $\Omega_\infty = -ak^2$
- Taub–NUT: "non-rigid rotation" with angular momentum distribution along the Misner string (vanishing integral) – asymptotically:
  - north pole: angular velocity $\Omega_\infty = nk^2$
  - south pole: no angular velocity

Pictorially: nuts and Misner strings

Figure: Kerr vs. Taub–NUT

How is Taub–NUT related to rotation?

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The Zermelo problem

What is the minimal-time trajectory of a non-relativistic ship sailing on a space with positive-definite metric $\mathrm{d}t^2 = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ *and velocity* $U^i = {}^{\mathrm{d}x^i}/_{\mathrm{d}t}$ *s.t.* $\|\mathbf{U}\|^2 = 1$*?*

- time functional is

$$T = \int \mathrm{d}t \sqrt{h_{ij} U^i U^j}$$

- minimization is realized with geodesics of $\mathrm{d}t^2$

What happens in the presence of a lateral drifting flow $\mathbf{W} = W^i \partial_i$ *("wind" or "tide")?* [Zermelo '31]

- velocity: $U^i = {}^{dx^i}\!/_{dt} = V^i + W^i$
  - $\mathbf{U}$: vector tangent to the trajectory
  - $\mathbf{V}$: "propelling" velocity with $\|\mathbf{V}\|^2 = 1$
    - no longer aligned with the trajectory
    - instantaneous navigation road – velocity of the ship with respect to a local frame dragged by the drifting flow
- norm: $\mathbf{U}^2 = 1 + \mathbf{W}^2 + 2\mathbf{V} \cdot \mathbf{W}$

- time functional is

$$\begin{aligned} T &= \int \mathrm{d}t \left(\sqrt{\frac{\mathbf{U}^2}{1-\mathbf{W}^2} + \left(\frac{\mathbf{W}\cdot\mathbf{U}}{1-\mathbf{W}^2} \right)^2} - \frac{\mathbf{W}\cdot\mathbf{U}}{1-\mathbf{W}^2} \right) \\ &= \int \mathrm{d}t \left(\sqrt{\left(\frac{h_{ij}}{\lambda} + \frac{W_i W_j}{\lambda^2} \right) U^i U^j} - \frac{W_k U^k}{\lambda} \right) \end{aligned}$$

  with $\lambda = 1 - \mathbf{W}^2$

- minimization is realized with null geodesics of the Zermelo metric

$$\mathrm{d}s^2 = \frac{1}{\lambda} \left(-\mathrm{d}t^2 + h_{ij} \left(\mathrm{d}x^i - W^i \mathrm{d}t \right) \left(\mathrm{d}x^j - W^j \mathrm{d}t \right) \right)$$

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Note: the time functional is of Randers type with Finsler Lagrangian

$$T = \int \mathrm{d}t \, F(x^i, U^i)$$

with

$$F(x^i, U^i) = \sqrt{a_{ij} U^i U^j} + b_i U^i$$

and

$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i W_j}{\lambda^2} \quad b_i = -\frac{h_{ij} W^j}{\lambda}$$

the data of the Randers form

Randers forms and Zermelo metrics [Zermelo '31, Randers '41]

The boundary geometries describing vorticity are stationary metrics of the Randers–Papapetrou form

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - b\right)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

Breaking of global hyperbolicity if $\exists x$ *s.t.* $b^2 > 1$ ($b^2 = a^{ij}b_i b_j$)

Potential closed time-like curves – not geodesics

- Kerr: globally hyperbolic
- Taub–NUT: $\exists$ CTCs
  - equivalent to studying charged particles on S^2 in a Dirac monopole background – QHE [Haldane '83]
  - horizon around the vortex – local thermodynamic equilibrium questionable in the *chronologically unprotected region*

Equivalently recast as Zermelo metrics $(a, b) \leftrightarrow (h, W)$

$$\mathrm{d}s^2 = \frac{1}{1 - W^2}\left(-\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i - W^i \mathrm{d}t\right)\left(\mathrm{d}x^j - W^j \mathrm{d}t\right)\right)$$

Analogue-gravity geometries originating from bulk solutions of Einstein's equations via holography

- Zermelo metrics are acoustic: null geodesics describe sound propagation in (non-)relativistic fluids moving on geometries $h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ with velocity field $\mathbf{W} = W^i\partial_i$ [see e.g. Visser '97]
- CTCs & horizons capture physical effects: sound propagation in supersonic-flow regions $(W^2 > 1)$

Similar approaches exist for light propagation in moving media such as (non-)relativistic (conformal) fluids

Analogue gravity picture

Zermelo metrics are acoustic *[see e.g. Visser '97, Chapline, Mazur '04]*

Propagation in $D-1$-dim moving media

$$\Updownarrow$$

Waves or rays in D-dim "analogue" curved space–times

$$\mathrm{d}s^2 = \frac{\varrho}{c_\mathrm{s}}\left(-c_\mathrm{s}^2\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i - W^i\mathrm{d}t\right)\left(\mathrm{d}x^j - W^j\mathrm{d}t\right)\right)$$

Null geodesics describe sound propagation in non-relativistic fluids moving on geometries $h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ *with velocity fields* $\mathbf{W} = W^i\partial_i$

- inviscid, isolated, barotropic ($\mathrm{d}h = {}^{\mathrm{d}p}/_{\varrho}$)
- local mass density ϱ and pressure p
- local sound velocity $c_\mathrm{s} = {}^1/_{\sqrt{\partial\rho/\partial p}}$

Alternatively the whole boundary set up could be a gravitational analogue of sound propagating in moving fluids or light in moving dielectrics – acoustic/optical black holes

As such our examples fall in a larger class of backgrounds studied in analogue systems [Gibbons et al. '08] – here equipped with a stress tensor

Randers & Zermelo backgrounds address the problems of

- motion of charged particles in magnetic fields
- sailing in the presence of a drift force
- sound propagation in moving media

and are dual to each other

Where are we?

Exploratory tour of some properties of conformal holographic fluids moving in non-trivial gravitational backgrounds

- inertial
- carrying vorticity

Vorticity appears in various fashions

- Kerr $\longrightarrow$ solid rotation on the boundary: dipole
- Taub–NUT $\longrightarrow$ vortex on the boundary: monopole

More general multipoles?

More general "multipolar" vortices on the boundary

$$b = 2(-1)^\ell \alpha \left(1 - P_\ell(\cos\vartheta)\right) \mathrm{d}\varphi$$
$$\omega = (-1)^\ell \alpha \, P'_\ell(\cos\vartheta) \sin\vartheta \, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi$$

- for odd ℓ there is indeed a vortex around the track of the Misner string at the south pole with a nut-like charge

$$\alpha = -\frac{1}{4\pi}\int \omega$$

- for even ℓ the Misner string does not reach the poles and the total charge vanishes – e.g. Kerr as a dipole with $\alpha = {}^a/_{3\Xi}$

Bulk realization for $\ell \geq 3$: generalization of Weyl multipoles [Weyl '19]
($\ell = 0$ *is Schwarzschild with* $\mathrm{d}t \to \mathrm{d}t + \mathrm{d}\varphi$) [work in progress]

Conformal fluids with vorticity

Class of bulk solutions describing conformal fluids in $2+1$ dim with vorticity – backgrounds still to be unravelled for $\ell \geq 3$ and most importantly perturbations to be understood *[see e.g. Bakas '08]*

- Spectrum of bulk excitations $\rightarrow$ *anyons* on the boundary – like in exotic BEC phases (under experimental investigation)
- Transport coefficients like shear viscosity

$$\eta \sim \frac{\varepsilon + p}{\Omega} = \frac{sT}{\Omega}$$

(reminiscent of response in magnetized plasmas)

Bonus: alternative analogue interpretation of the boundary theories

propagation of sound/light in moving media (Randers vs. Zermelo)

More ambitious

Recast the superfluid phase transition and the appearance of vortices

Combine Kerr and nut charge in AdS Kerr Taub–NUT

- add a $U(1)$ and a scalar field (order parameter)
- analyse the phase diagramme (M temperature, $\{a, n\}$ rotation)
- study the formation of a vortex as nut–anti-nut dissociation

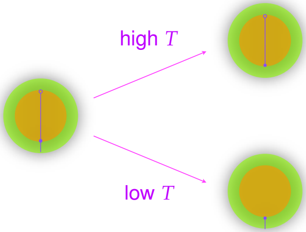

Figure: high-T vs. low-T stable phase

Bonus

Alternative analogue interpretation of the boundary backgrounds: propagation of sound/light in moving media (Randers & Zermelo)

- provides holographic AdS/analogue-gravity correspondence
- evades the CTCs caveats within supersonic/superluminal flows

Bulk for general Randers–Papaterou geometries?