KOUNNASFest

Cyprus, Sept. 29, 2012

HAPPY BIRTHDAY COSTA

GAUGE THEORIES

AND

NON-COMMUTATIVE GEOMETRY

Kounnasfest

Motivation

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4) Large N gauge theories and matrix models.

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3) Seiberg-Witten map.

4) Large N gauge theories and matrix models.

5) The construction of gauge theories using the techniques of non-commutative geometry.

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}$$

simplest case: θ is constant (canonical, or Heisenberg case).

Other cases:

$[x_\mu, x_\nu] = iF^{\rho}_{\mu\nu} x_\rho$ (Lie algebra case)

$x_\mu x_\nu = q^{-1} R^{\rho\sigma}_{\mu\nu} x_\rho x_\sigma$ (quantum space case)

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Definition of the derivative:

$\partial^\mu x_\nu = \delta^\mu_\nu \qquad [x_\mu, f(x)] = i\theta_{\mu\nu}\partial^\nu f(x)$

Define a * product

$f * g = e^{\frac{i}{2}\frac{\partial}{x_\mu}\theta_{\mu\nu}\frac{\partial}{y_\nu}} f(x)g(y)|_{x=y}$

All computations can be viewed as expansions in θ

expansions in the external field

More efficient ways?

Large N field theories

$\phi^i(x)\ i = 1, ..., N$; $N \to \infty$

$\phi^i(x) \to \phi(\sigma, x)\ 0 \leq \sigma \leq 2\pi$

$\sum_{i=1}^{\infty} \phi^i(x)\phi^i(x) \to \int_0^{2\pi} d\sigma(\phi(\sigma, x))^2$

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however

For a Yang-Mills theory, the resulting expression is local

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there exists a large N limit such that:

$$(A_\mu(x))^a_b \to \mathcal{A}_\mu(x,\sigma_1,\sigma_2) \qquad\qquad (F_{\mu\nu}(x))^a_b \to \mathcal{F}_{\mu\nu}(x,\sigma_1,\sigma_2) \quad (1)$$

Gauge theories on surfaces

The gauge transformations of the $SU(N)$ Yang-Mills theory become area preserving diffeomorphisms of the surface:

$$\delta A_\mu = \partial_\mu \omega(x) + [A_\mu, \omega] \;\rightarrow\; \delta \mathcal{A}_\mu = \partial_\mu \omega(x, \sigma_1, \sigma_2) + \{\mathcal{A}_\mu, \omega\}$$
$$\delta F_{\mu\nu} = [F_{\mu\nu}, \omega] \;\rightarrow\; \delta \mathcal{F}_{\mu\nu} = \{\mathcal{F}_{\mu\nu}, \omega\}$$
$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + \{\mathcal{A}_\mu, \mathcal{A}_\nu\} \tag{2}$$

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The classical action becomes

$$S \ \sim \ -\frac{1}{4} \int Tr F_{\mu\nu} F^{\mu\nu} d^4x \ \rightarrow \ \mathcal{S} \ \sim \ \frac{1}{4} \int \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} d^4x d\sigma_1 d\sigma_2 \tag{5}$$

Gauge theories on surfaces

The $SU(N)$ algebra $\rightarrow$ The algebra of the area preserving diffeomorphisms of a closed surface.

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-The structure constants of $[SDiff(S^2)]$ are the limits for large N of those of $SU(N)$.

Gauge theories on surfaces

For a sphere:

$$x_1 = cos\phi\ sin\theta, \quad x_2 = sin\phi\ sin\theta, \quad x_3 = cos\theta$$

$$Y_{l,m}(\theta,\phi) = \quad \sum_{\substack{i_k=1,2,3\\ k=1,\ldots,l}} \alpha^{(m)}_{i_1 \ldots i_l}\ x_{i_1} \ldots x_{i_l}$$

where $\alpha^{(m)}_{i_1 \ldots i_l}$ is a symmetric and traceless tensor.
For fixed l there are $2l+1$ linearly independent tensors $\alpha^{(m)}_{i_1 \ldots i_l}$, $m = -l, ..., l$.

Choose, inside $SU(N)$, an $SU(2)$ subgroup.

$$[S_i, S_j] = i\epsilon_{ijk} S_k$$

A basis for $SU(N)$:

$$S^{(N)}_{l,m} = \quad \sum_{\substack{i_k=1,2,3 \\ k=1,\ldots,l}} \alpha^{(m)}_{i_1 \ldots i_l} \; S_{i_1} \ldots S_{i_l}$$

$$[S^{(N)}_{l,m}, \; S^{(N)}_{l',m'}] = i f^{(N) l'', m''}_{l,m;\; l',m'} \; S^{(N)}_{l'',m''}$$

The three $SU(2)$ generators S_i, rescaled by a factor proportional to $1/N$, will have well-defined limits as N goes to infinity.

$$S_i \rightarrow T_i = \frac{2}{N} S_i$$
$$[T_i, T_j] = \frac{2i}{N} \epsilon_{ijk} T_k$$
$$T^2 = T_1^2 + T_2^2 + T_3^2 = 1 - \frac{1}{N^2}$$

In other words: under the norm $\|x\|^2 = Trx^2$, the limits as N goes to infinity of the generators T_i are three objects x_i which commute and are constrained by

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$\frac{N}{2i}\ [f,g] \rightarrow \quad \epsilon_{ijk}\ x_i\ \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_k}$$

$$\frac{N}{2i}\ [T^{(N)}_{l,m}, T^{(N)}_{l',m'}] \rightarrow\ \{Y_{l,m}, Y_{l',m'}\}$$

$$N[A_\mu, A_\nu] \rightarrow \quad \{A_\mu(x,\theta,\phi), A_\nu(x,\theta,\phi)\}$$

II. To all orders

Given an $SU(N)$ Yang-Mills theory in a $d-$dimensional space

$$A_\mu(x) \;=\; A_\mu^a(x)\, t_a$$

there exists a reformulation in d+2 dimensions

$$A_\mu(x) \to \mathcal{A}_\mu(x, z_1, z_2) \qquad F_{\mu\nu}(x) \to \mathcal{F}_{\mu\nu}(x, z_1, z_2)$$

with $$[z_1, z_2] = \frac{2i}{N}$$

$$[A_\mu(x), A_\nu(x)] \to \{\mathcal{A}_\mu(x, z_1, z_2), \mathcal{A}_\nu(x, z_1, z_2)\}_{Moyal}$$

$$[A_\mu(x), \Omega(x)] \to \{\mathcal{A}_\mu(x, z_1, z_2), \mathit{\Omega}(x, z_1, z_2)\}_{Moyal}$$

$$\int d^4x\ Tr\left(F_{\mu\nu}(x) F^{\mu\nu}(x)\right) \quad \to$$
$$\int d^4x dz_1 dz_2\ \mathcal{F}_{\mu\nu}(x, z_1, z_2) * \mathcal{F}^{\mu\nu}(x, z_1, z_2)$$

For the sphere:
We can parametrize the T_i's in terms of two operators, z_1 and z_2.

$$T_+ = T_1 + iT_2 = e^{\frac{iz_1}{2}}\ (1 - z_2^2)^{\frac{1}{2}}\ e^{\frac{iz_1}{2}}$$
$$T_- = T_1 - iT_2 = e^{-\frac{iz_1}{2}}\ (1 - z_2^2)^{\frac{1}{2}}\ e^{-\frac{iz_1}{2}}$$
$$T_3 = z_2$$

If we assume that z_1 and z_2 satisfy:

$[z_1, z_2] = \frac{2i}{N}$

The T_i's satisfy the $SU(2)$ algebra.

If we assume that the T_i's satisfy the $SU(2)$ algebra, the z_i's satisfy the Heisenberg algebra

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Gauge transformations are:

-Diffeomorphisms *space-time*

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Question: Is there a space on which Internal symmetry transformations act as Diffeomorphisms?

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Question: Is there a space on which Internal symmetry transformations act as Diffeomorphisms?

Answer: Yes, but it is a space with non-commutative geometry.
A space defined by an algebra of matrix-valued functions

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Hopefully, a subject for Costas' 70eth birthday

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