A new look at One-Loop Amplitudes in String Theory

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Based on work with
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ΚΟΥΝΝΑΣ - *fest !*

Cyprus, 29 September 2012

MAX-PLANCK-GESELLSCHAFT

Closed String perturbation theory

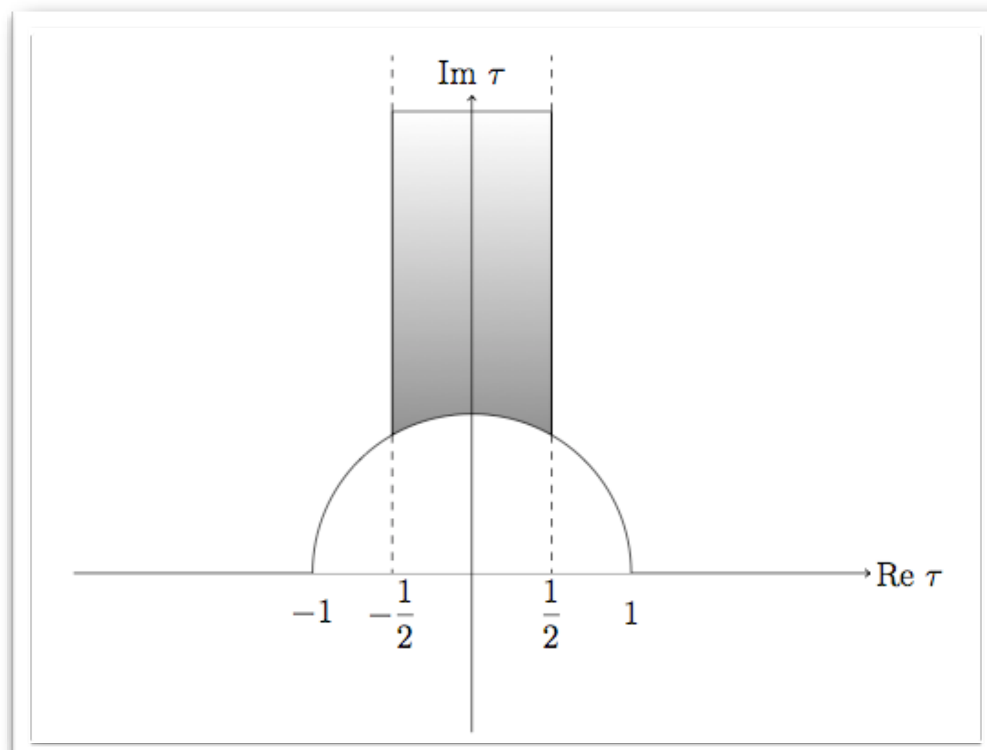
Topological expansion over closed Riemann surfaces

$$\sum_{g=0}^{\infty} g_s^{2(g-1)} \int_{\text{moduli}} \int \mathcal{D}g_{ab}\mathcal{D}X\,\mathcal{D}\psi\,\ldots\mathcal{V}_i(z_i)\ldots\,e^{-S[X,\psi,g_{ab},\ldots]}$$

$g=1$: Torus Amplitude

$$\int_{\mathcal{F}} d\mu\;\mathcal{A}(\tau,\bar{\tau})$$

- Complex structure of worldsheet torus $\tau\in\mathcal{H}$
- Gauge modular group of large diffeomorphisms $PSL(2;\mathbb{Z})$
- Integration restricted over fundamental domain $\mathcal{F}=\{\tau\in\mathcal{H}\;:\;|\tau|\geq 1\;,\;|\tau_1|\leq 1/2\}$
- Invariant measure $d\mu := \frac{d^2\tau}{\tau_2^2}$
- $\mathcal{A}(\tau,\bar{\tau})$ modular invariant function



$\alpha' = 1$

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Some common examples

$$\int_{\mathcal{F}} d\mu \; \Gamma_{(d+k,d)}(G,B,Y;\tau_1,\tau_2)\,\Phi(\tau)$$

- Gauge threshold corrections $R^2 F^{2h-2}$ in heterotic on $K3 \times T^2$
- F^4 couplings in heterotic on T^d
- R^4 couplings in type II on T^d
- R^2 couplings in type II on $K3 \times T^2$
- ...

Dixon, Kaplunovski, Louis ; Harvey, Moore
Bachas, Fabre, Kiritsis, Obers, Vanhove
Green, Vanhove, Kiritsis, Pioline
Gregori, Kiritsis, Kounnas, Obers, Petropoulos, Pioline

Physical interest

- Stringy correction to one-loop amplitudes : massive string states running in the loop
- For special vacua and for special classes of interaction, perturbative corrections stop at torus amplitude : test string dualities (BPS-saturated couplings, F-terms, topological amplitudes)
- Spontaneously broken SUSY : perturbative corrections to effective potential
- Superstrings at finite temperature : effective thermal potential

1-loop effective potential at points of extended symmetry : long-standing puzzles in string thermodynamics and string cosmology

Hagedorn phase transition

Initial Singularity

C. Angelantonj, M. Cardella, N. Irges 2006
C. Angelantonj, C. Kounnas, H. Partouche, N. Toumbas 2009
I.F., C. Kounnas 2009
I.F., C. Kounnas, N. Toumbas 2010
I.F., C. Kounnas, H. Partouche, N. Toumbas 2010

The problem at hand

In many string theory applications, one encounters modular integrals of the form

$$I = \int_{\mathcal{F}} d\mu\ \Gamma_{(d+k,d)}(G,B,Y)\ \Phi(\tau)$$

Such integrals appear naturally in one-loop corrections to certain BPS-saturated couplings in the low energy effective action of Heterotic or Type II superstrings

$\Gamma_{(d+k,d)}(G,B,Y)$ Narain lattice of signature (*d*+*k*,*d*) depending on compactification moduli in

$$\frac{SO(d+k,d)}{SO(d+k)\times SO(d)}$$

$\Phi(\tau)$ is a weak almost holomorphic modular form of negative weight $w=-k/2$ and has (at most) a simple pole in $q=\exp(2\pi i\tau)$ at the cusp

$$\Phi(\tau) = \sum_{\substack{2n_1+4n_2+6n_3=12+w \\ n_i\geq 0}} c_{n_1,n_2,n_3}\ \frac{\hat{E}_2^{n_1} E_4^{n_2} E_6^{n_3}}{\Delta}$$

The major difficulty with evaluating this integral is the unwieldy shape of $\mathcal{F}$

The known way out is a procedure that goes by the name "orbit method" or simply "unfolding"

The orbit method

Start from $\int_{\mathcal{F}} d\mu \; f(\tau, \bar{\tau})$ with f being a modular function

Express f as a sum over modular orbits (Poincaré series representation)

$$f(\tau, \bar{\tau}) = \tfrac{1}{2} \sum_{\gamma \in SL(2;\mathbb{Z})/\Gamma_\infty} \varphi(\gamma \cdot \tau, \gamma \cdot \bar{\tau})$$

$$\gamma \cdot \tau \equiv \frac{a\tau + b}{c\tau + d}$$

φ is called the "seed" and is assumed invariant under rigid translations

$$\Gamma_\infty = \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \subset SL(2;\mathbb{Z})$$

Plug it into the integral and change variables $\tau' = \gamma \cdot \tau$

$$\tfrac{1}{2} \sum_{\gamma \in SL(2;\mathbb{Z})/\Gamma_\infty} \int_{\mathcal{F}} d\mu \; \varphi(\gamma \cdot \tau, \gamma \cdot \bar{\tau}) = \tfrac{1}{2} \sum_{\gamma \in SL(2;\mathbb{Z})/\Gamma_\infty} \int_{\gamma\mathcal{F}} d\mu \; \varphi(\tau', \bar{\tau}')$$

Summing over SL(2;Z)-orbits, the fundamental domain is "unfolded" to the half-infinite strip

$$\longrightarrow \int_{\mathcal{H}/\Gamma_\infty} d\mu \; \varphi(\tau, \bar{\tau})$$

$$\mathcal{H}/\Gamma_\infty \equiv \{0 < \tau_2 < \infty, -\tfrac{1}{2} \le \tau_1 < \tfrac{1}{2}\}$$

- τ_1 : imposes level matching
- τ_2 : Schwinger representation

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Traditional unfolding against the lattice

Traditionally : use orbit decomposition of the lattice (in *Lagrangian* rep.)

$$\Gamma_{(1,1)}(R) = R \sum_{\tilde{m},n\in\mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2}|\tilde{m}+\tau n|^2}$$

Extract $(m,n)=(0,0)$ orbit and factor out g.c.d. of non-zero windings $N=(m,n)$

$m=Np$, $n=Nq$ with $(p,q)=1$

$$\Gamma_{(1,1)}(R) = R + R \sum_{N\geq 1} \sum_{(p,q)=1} e^{-\frac{\pi (NR)^2}{\tau_2}|p+\tau q|^2}$$

$$\mathrm{Im}(\gamma\cdot\tau) = \frac{\tau_2}{|p+\tau q|^2}$$

$$\gamma = \begin{pmatrix} * & * \\ q & p \end{pmatrix}$$

Poincaré series with seed $\varphi(\tau,\bar{\tau}) = \exp\left(-\frac{\pi (NR)^2}{\tau_2}\right)$

$$\int_{\mathcal{F}} d\mu\ \Gamma_{(1,1)}(R)\ j(\tau) = R\int_{\mathcal{F}} j(\tau) + 2R\sum_{N\geq 1}\int_0^\infty \frac{d\tau_2}{\tau_2^2}\ e^{-\frac{\pi (NR)^2}{\tau_2}}\ j_0(\tau_2)$$

$$j_0(\tau_2) = \int_{-1/2}^{1/2} d\tau_1\ j(\tau) = 0$$

$$\int_{\mathcal{F}} d\mu\ \Gamma_{(1,1)}(R)\ j(\tau) = R\int_{\mathcal{F}} d\mu\ j(\tau) = -8\pi R$$

NOT invariant under T-duality !

Absolute convergence: interchange integration with summation

UV : guaranteed by lattice

IR : "extra" massless modes (at T-self-dual point)

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Traditional unfolding against the lattice

Unfolding against the lattice is useful for extracting the large volume behaviour of the amplitude

$R \gg 1 \quad , \quad R \ll 1$

Loss of absolute convergence around extended symmetry points (fixed points under T-duality) obscures the behaviour of the amplitude in these regions

T-duality symmetry is not manifest in this representation

- Unfolding against the lattice starts in Lagrangian representation
- Winding sum is decomposed into SL(2;Z) orbits
- Each distinct orbit is used separately to unfold the fundamental domain

example:

$$\int_{\mathcal{F}} d\mu \; \Gamma_{(2,2)}(T,U) = -\log\Big(T_2 U_2 |\eta(T)\eta(U)|^4\Big) + \text{const.}$$

(after subtraction of IR divergent piece)

Dixon, Kaplunovsky, Louis 1991

- Zero orbit $\frac{\pi}{3}T_2$
- Non-degenerate orbit $-\frac{\pi}{3}T_2 - \log|\eta(T)|^4$
- Degenerate orbit (strip) $-\log\Big(T_2 U_2 |\eta(U)|^4\Big)$

!

NONE of the individual pieces is invariant under T-duality
$SL(2;\mathbb{Z})_T \times SL(2;\mathbb{Z})_U \times \mathbb{Z}_2$

Traditional unfolding against the lattice

A more complicated example:

$$\int_{\mathcal{F}} d\mu\, \Gamma_{2,2}(T,U)\, \frac{\hat{E}_2\, E_4\, E_6}{\Delta} \simeq \mathrm{Re}\Bigg[-24 \sum_{k>0} \left(11\, \mathrm{Li}_1(e^{2\pi i k T}) - \frac{30}{\pi T_2\, U_2} \mathcal{P}(kT) \right)$$

$$- 24 \sum_{\ell>0} \left(11\, \mathrm{Li}_1(e^{2\pi i \ell U}) - \frac{30}{\pi T_2\, U_2} \mathcal{P}(\ell U) \right)$$

$$+ \sum_{k>0,\ell>0} \left(\tilde{c}(k\ell)\, \mathrm{Li}_1(e^{2\pi i (kT+\ell U)}) - \frac{3\, c(k\ell)}{\pi T_2\, U_2} \mathcal{P}(kT+\ell U) \right)$$

$$+ \mathrm{Li}_1(e^{2\pi i (T_1 - U_1 + i|T_2 - U_2|)}) - \frac{3}{\pi T_2\, U_2} \mathcal{P}\left(T_1 - U_1 + i|T_2 - U_2|\right) \Bigg]$$

$$+ \frac{60\, \zeta(3)}{\pi^2\, T_2 U_2} + 22 \log\left(\frac{8\pi e^{1-\gamma}}{\sqrt{27}}\, T_2 U_2 \right)$$

$$+ \left(\frac{4\pi}{3} \frac{U_2^2}{T_2} - \frac{22\pi}{3} U_2 - 4\pi T_2 \right) \Theta(T_2 - U_2)$$

$$+ \left(\frac{4\pi}{3} \frac{T_2^2}{U_2} - \frac{22\pi}{3} T_2 - 4\pi U_2 \right) \Theta(U_2 - T_2)$$

where $\mathcal{P}(z) = y \mathrm{Li}_2(e^{2\pi i z}) + \frac{1}{2\pi} \mathrm{Li}_3(e^{2\pi i z})$

- Result is chamber dependent
- Obscures singularities of the amplitude !
- Hard to check T-duality invariance !
- Useful for extracting asympotic behaviour in large volume limit

A modular invariant regulator

Modular invariant way of regularizing on-shell IR divergences

E. Kiritsis, C. Kounnas 1995

Curve 4d background so that the string spectrum acquires a mass gap

$$\mu = \sqrt{\frac{1}{k+2}}$$

$SU(2)_k$ WZW model + scalar with background charge $Q^2=2/(k+2)$

$$\int_{\mathcal{F}} d\mu \; Z(\tau,\bar{\tau}) \; \Gamma(\mu)$$

$$\Gamma(\mu) = 4\sqrt{x}\,\frac{\partial}{\partial x}\left[\Gamma_{(1,1)}(\sqrt{x}) - \Gamma_{(1,1)}(\sqrt{x}/4)\right]\Big|_{x=k+2}$$

Idea : Let's unfold against something else !

Goal: find some other way to unfold that does not spoil the manifest T-duality symmetries of the lattice

Look for representation that captures the behaviour around T-self-dual points

1 $$\int_{\mathcal{F}} d\mu\ \Gamma_{d,d}(G,B;\tau,\bar\tau)$$ Rankin-Selberg-Zagier method

↓ deform

$$\int_{\mathcal{F}_T} d\mu\ \Gamma_{d,d}(G,B;\tau,\bar\tau)\,E^\star(s;\tau)$$ —unfold→ $$\int_0^\infty d\tau_2\ \tau_2^{s-2}\int_{-1/2}^{1/2} d\tau_1\ \tau_2^{d/2}\sum_{m,n} e^{-2\pi\tau_2\mathcal{M}^2}\,e^{2\pi i\tau_1 m^T n}$$

$$E^\star(s;\tau)=\zeta^\star(2s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\Big[\mathrm{Im}(\gamma\cdot\tau)\Big]^s$$

↓ extract residue

$$\mathrm{Res}_{s=1}E^\star(s;\tau)=\frac{1}{2}$$

$$\frac{1}{2}\int_{\mathcal{F}} d\mu\ \Gamma_{d,d}(G,B;\tau,\bar\tau)=\mathrm{Res}_{s=1}\left[\int_0^\infty d\tau_2\ \tau_2^{s-2+d/2}\sum_{m^T n=0} e^{-2\pi\tau_2\mathcal{M}^2}\right]$$

- Manifestly T-duality invariant !
- For DKL integral : gives the answer in a few lines !
- No need for delicate regularization of degenerate orbit

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New method required !

2 $$\int_{\mathcal{F}} d\mu\ \Gamma_{d+k,d}(G,B,Y;\tau,\bar{\tau})\,\Phi(\tau)$$ unfold the elliptic genus !

What happens for integrands which are of rapid growth at the cusp ?
(unphysical tachyon)

A new method: unfolding against the elliptic genus !

We need a Poincaré representation of modular form Φ

But which is the correct seed ?

- Hyperbolic Laplacian Δ acts as Casimir operator on modular forms Φ

$$\Delta_w = 2\tau_2^2\,\partial_{\bar\tau}\left(\partial_\tau - \frac{iw}{2\tau_2}\right)$$

- Construct Φ by Poincaré representation such that the seed f is an eigenmode of Δ

Impose order-κ pole at the cusp $\Phi \sim q^{-\kappa} + \dots$

The Poincaré series must be absolutely convergent (for w<0) to justify the unfolding

regularization ?

These conditions lead to the seed $\varphi(\tau,\bar\tau) = \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}$

$$\mathcal{M}_{s,w}(y) = |4\pi y|^{-w/2} M_{\frac{w}{2}\mathrm{sgn}(y),\, s-\frac{1}{2}}(4\pi|y|)$$

Whittaker M-function

$$M_{\lambda,\mu}(z) = e^{-z/2}\, z^{\mu+\frac{1}{2}}\ {}_1F_1(\mu-\lambda+\tfrac{1}{2}; 1+2\mu; z)$$

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A new Poincaré series

D. Niebur 1973
D. Hejhal 1983
J. Bruinier 2002

This seed defines the Niebur-Poincaré series

$$\mathcal{F}(s,\kappa,w)=\tfrac{1}{2}\sum_{\gamma\in SL(2;\mathbb{Z})/\Gamma_\infty}(c\tau+d)^{-w}\,\mathcal{M}_{s,w}(-\kappa\,\mathrm{Im}\,\gamma\cdot\tau)\;e^{-2\pi i\kappa\mathrm{Re}(\gamma\cdot\tau_1)}$$

$$=\tfrac{1}{2}\sum_{(c,d)=1}(c\tau+d)^{-w}\,\mathcal{M}_{s,w}\left(-\frac{\kappa\tau_2}{|c\tau+d|^2}\right)\;\exp\left\{-2\pi i\kappa\left(\frac{a}{c}-\frac{c\tau_1+d}{c|c\tau+d|^2}\right)\right\}$$

- Converges absolutely for Re(s)>1
- For $\kappa>0$, correct behaviour at the cusp $\quad \mathcal{M}_{s,w}(-\kappa\tau_2)e^{-2\pi i\kappa\tau_1}\sim\dfrac{\Gamma(2s)}{\Gamma(s+\frac{w}{2})}\,q^{-\kappa}$
- By construction : eigenmode of the hyperbolic Laplacian

$$\left[\Delta_w+\frac{s(1-s)}{2}+\frac{w(w+2)}{8}\right]\mathcal{F}(s,\kappa,w)=0$$

Spectrum is obtained by studying Fourier expansion & using raising and lowering operators

$$D_w=\frac{i}{\pi}\left(\partial_\tau-\frac{iw}{2\tau_2}\right)\qquad D_w\cdot\mathcal{F}(s,\kappa,w)=2\kappa(s+\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w+2)$$

$$\bar{D}_w=-i\pi\tau_2^2\,\partial_{\bar\tau}\qquad \bar{D}_w\cdot\mathcal{F}(s,\kappa,w)=\frac{1}{8\kappa}(s-\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w-2)$$

In string theory, at most κ=1

A new Poincaré series

$$\left[\Delta_w + \frac{s(1-s)}{2} + \frac{w(w+2)}{8}\right]\mathcal{F}(s,\kappa,w) = 0$$

Weak quasi-holomorphic modular forms are eigenmodes of the Laplacian with eigenvalue *-w/2*

The N-P series has the same eigenvalue for *s=1-w/2*

In general, the N-P series with *s=1-w/2* is a (weak) harmonic Maass form (Mock + Shadow)

However, by taking linear combinations of N-P series with definite coefficients, the Shadows cancel and the linear combination represents any weak holomorphic modular form !

Weak quasi-holomorphic modular forms can be formed from linear combinations of N-P series with *s=1-w/2+n*

The spectrum of modular forms as limits of the N-P series

Theorem

All weak almost holomorphic modular forms can be expressed as linear combinations of absolutely convergent Niebur-Poincaré series

$w=0$

$$\frac{\hat{E}_2 E_4 E_6}{\Delta} = \mathcal{F}(2,1,0) - 5\,\mathcal{F}(1,1,0) - 144$$
$$\frac{\hat{E}_2^2 E_4^2}{\Delta} = \frac{1}{5}\,\mathcal{F}(3,1,0) - 4\mathcal{F}(2,1,0) + 13\,\mathcal{F}(1,1,0) + 144$$
$$\frac{\hat{E}_2^3 E_6}{\Delta} = \frac{3}{175}\,\mathcal{F}(4,1,0) - \frac{3}{5}\,\mathcal{F}(3,1,0) + \frac{33}{5}\,\mathcal{F}(2,1,0) - 17\,\mathcal{F}(1,1,0) - 144$$
$$\frac{\hat{E}_2^4 E_4}{\Delta} = \frac{1}{1225}\,\mathcal{F}(5,1,0) - \frac{6}{175}\,\mathcal{F}(4,1,0) + \frac{18}{35}\,\mathcal{F}(3,1,0) - \frac{16}{5}\,\mathcal{F}(2,1,0) + \frac{29}{5}\,\mathcal{F}(1,1,0) + \frac{144}{5}$$

$$w=0$$

$$\frac{\hat{E}_2 E_4 E_6}{\Delta} = \mathcal{F}(2,1,0) - 5\,\mathcal{F}(1,1,0) - 144$$
$$\frac{\hat{E}_2^2 E_4^2}{\Delta} = \frac{1}{5}\,\mathcal{F}(3,1,0) - 4\mathcal{F}(2,1,0) + 13\,\mathcal{F}(1,1,0) + 144$$
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$$\frac{\hat{E}_2^6}{\Delta} = \frac{1}{1926925}\mathcal{F}(7,1,0) - \frac{3}{2695}\mathcal{F}(5,1,0) + \frac{6}{175}\mathcal{F}(4,1,0) - \frac{3}{7}\mathcal{F}(3,1,0) + \frac{12}{5}\mathcal{F}(2,1,0) - \frac{29}{7}\mathcal{F}(1,1,0) - \frac{144}{7}$$

$$w=-8$$

$$\frac{\hat{E}_2^2}{\Delta} = \frac{1}{2854051200}\mathcal{F}(7,1,-8) - \frac{1}{3991680}\mathcal{F}(5,1,-8)$$

$$w=-10$$

$$\frac{\hat{E}_2}{\Delta} = \frac{1}{6277020800}\mathcal{F}(7,1,-10)$$

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Unfolding against the N-P series gives a BPS sum

- τ_1-integration : picks BPS state contribution
- τ_2-integration : Schwinger representation

$$R.N. \int_F d\mu\, \Gamma_{(d+k,d)}\ \mathcal{F}(s,\kappa,-\tfrac{k}{2}) = \lim_{T\to\infty}\left[\int_{\mathcal{F}_T} d\mu\, \Gamma_{(d+k,d)}\ \mathcal{F}(s,\kappa,-\tfrac{k}{2}) + f_0(s)\,\frac{T^{\frac{d}{2}+\frac{k}{4}-s}}{s-\frac{d}{2}-\frac{k}{4}}\right]$$

$$= \int_0^\infty d\tau_2\ \tau_2^{d/2-2}\ \mathcal{M}_{s,-\frac{k}{2}}(-\kappa\tau_2)\ \sum_{\rm BPS}\ e^{-\pi\tau_2\,(P_L^2+P_R^2)/2}$$

for generic values of $s \neq \frac{d}{2}+\frac{k}{4}$

$$I = (4\pi\kappa)^{1-\frac{d}{2}}\,\Gamma(s+\tfrac{d}{2}+\tfrac{k}{4}-1)$$
$$\times \sum_{\rm BPS} {}_2F_1\left(s-\frac{k}{4}, s+\frac{d}{2}+\frac{k}{4}-1; 2s; \frac{4\kappa}{P_L^2}\right)\left(\frac{P_L^2}{4\kappa}\right)^{1-s-\frac{d}{2}-\frac{k}{4}}$$

- For Re(s)>$d/2+k/4$, sum converges absolutely, with a simple pole at s=$d/2+k/4$
- Manifestly T-duality invariant
- Chamber independent

BPS state sums & Singularity Structure

$$n = s + \frac{w}{2} - 1$$

One-dimensional lattice

$$\int_{\mathcal{F}} d\mu \; \Gamma_{(1,1)}(R) \; \mathcal{F}(1+n,1,0) = 2^{2+2n} \sqrt{\pi}\, \Gamma(n+\tfrac{1}{2}) \left(R^{1+2n} + \frac{1}{R^{1+2n}} - \left| R^{1+2n} - \frac{1}{R^{1+2n}} \right| \right)$$

BPS state sums & Singularity Structure

$$n = s + \frac{w}{2} - 1$$

General result for $n>d/2-1$ or for odd-dimension (independently of n):

$$I_1 = (4\pi\kappa)^{1-\frac{d}{2}} \frac{\Gamma(2n+2+\frac{k}{2})\,\Gamma(n+\frac{d+k}{2})}{n!} \sum_{m=0}^{d/2-2} \binom{n}{m} \frac{(-)^m}{\Gamma(n-m+\frac{d+k}{2})}$$
$$\times \sum_{\text{BPS}} \left(\frac{P_L^2}{4\kappa}\right)^{n-m} \left[\Gamma(\tfrac{d}{2}-m-1) \left(\frac{P_R^2}{4\kappa}\right)^{m+1-\frac{d}{2}} - \sum_{\ell=0}^{2n+k/2} \frac{\Gamma(\frac{d}{2}-m-1+\ell)}{\ell!} \left(\frac{P_L^2}{4\kappa}\right)^{1+m-\frac{d}{2}-\ell} \right]$$

General result for even-dimension and $n \leq d/2-1$ is given by adding I_1+I_2, where:

$$I_2 = (4\pi\kappa)^{1-\frac{d}{2}} \frac{\Gamma(2n+2+\frac{k}{2})\,\Gamma(n+\frac{d+k}{2})}{n!} \sum_{\text{BPS}} \sum_{m=d/2-1}^{n} \binom{n}{m} \frac{(-)^m}{\Gamma(n-m+\frac{d+k}{2})} \left(\frac{P_L^2}{4\kappa}\right)^{n-m}$$
$$\times \left\{ -\sum_{\ell=m+2-d/2}^{2n+k/2} \frac{\Gamma(\frac{d}{2}-m-1+\ell)}{\ell!} \left(\frac{P_L^2}{4\kappa}\right)^{1+m-\frac{d}{2}-\ell} + \frac{(-)^{m+1-\frac{d}{2}}}{\Gamma(m+2-\frac{d}{2})} \left(\frac{P_R^2}{4\kappa}\right)^{m+1-\frac{d}{2}} \right.$$
$$\left. \times \left[H_{m+1-\frac{d}{2}} - \log\left(\frac{P_R^2}{P_L^2}\right) \right] - \frac{1}{\Gamma(m+2-\frac{d}{2})} \sum_{\ell=0}^{m+1-d/2} \binom{m+1-\frac{d}{2}}{\ell} \left(-\frac{P_L^2}{4\kappa}\right)^{m+1-\frac{d}{2}-\ell} H_{m+1-\frac{d}{2}-\ell} \right\}$$

BPS state sums & Singularity Structure

This is the appropriate representation to read-off the singularity structure of the integral around extended symmetry points

Extra massless states at $P_R=0$

In odd dimensions

- The integral always develops conical singularities
- For $d\geq 3$ real singularities appear from terms with $m<d/2-1$

In even dimensions

- Conical singularities never appear
- Real singularities always appear
- Power-like singularities in I_1 whenever $d\geq 4$
- Logarithmic singularities in I_2 for any (even) $d\leq 2n+2$

Technically singularities appear due to the unphysical tachyon contribution

Amplitudes involving linear combinations of modular forms, such that the unphysical tachyon pole is cancelled are regular at any point in Narain moduli space

Universal singularity behaviour in 2d

$$I_{2,2}(s=1+n,\kappa=1)\ \sim\ -\frac{(2n+1)!}{n!}\ \log|j(T)-j(U)|^4$$

Example of Gauge Threshold calculations

$\mathcal{N} = 2$ heterotic vacuum at the orbifold point $T^2 \times T^4/\mathbb{Z}_2$

In the absence of Wilson lines

$$E_8 \times E_8 \to E_8 \times E_7 \times SU(2)$$

BPS constraint

$$\tfrac{1}{4}P_L^2 - \tfrac{1}{4}P_R^2 = 1 \quad \leftrightarrow \quad m_i\, n^i = 1$$

Example of Gauge Threshold calculations

Without Wilson lines:

$$\Delta_{E_8} = -\frac{1}{12}\int_{\mathcal{F}} d\mu\ \Gamma_{(2,2)}(T,U)\ \frac{\hat{E}_2 E_4 E_6 - E_6^2}{\Delta} = \sum_{BPS}\left[1 + \frac{P_R^2}{4}\log\left(\frac{P_R^2}{P_L^2}\right)\right] + 72\log\Big(T_2 U_2 |\eta(T)\eta(U)|^4\Big) + \text{cte.}$$

$$\Delta_{E_7} = -\frac{1}{12}\int_{\mathcal{F}} d\mu\ \Gamma_{(2,2)}(T,U)\ \frac{\hat{E}_2 E_4 E_6 - E_4^3}{\Delta} = \sum_{BPS}\left[1 + \frac{P_R^2}{4}\log\left(\frac{P_R^2}{P_L^2}\right)\right] - 72\log\Big(T_2 U_2 |\eta(T)\eta(U)|^4\Big) + \text{cte.}$$

Now turn on Wilson lines - Higgs the E_8 group factor to its Coulomb branch:

$$\Delta_{E_7} = -\frac{1}{12}\int_{\mathcal{F}} d\mu\ \Gamma_{(2,10)}\ \frac{\hat{E}_2 E_6 - E_4^2}{\Delta} = \sum_{BPS}\left[1 + \frac{P_R^2}{4}\log\left(\frac{P_R^2}{P_L^2}\right) - \frac{2}{P_L^2} - \frac{8}{3\,P_L^4} - \frac{16}{3\,P_L^6} - \frac{64}{5\,P_L^8}\right]$$

Left- & right- moving momenta also depend on the Wilson lines Y and the BPS constraint now contains the U(1) charge vectors Q in the Cartan of E_8

$$m^T n + \tfrac{1}{2}\, Q^T Q = 1$$

Results regular at any point in moduli space and in any chamber !

One-loop BPS amplitudes with momentum insertions

Consider modular integrals with insertions of left/right- moving lattice momenta:

$$\int_{\mathcal{F}} d\mu \left[\tau_2^{-\lambda/2} \sum_{P_L,P_R} \rho(P_L\sqrt{\tau_2}, P_R\sqrt{\tau_2})\; q^{\frac{1}{4}P_L^2}\, \bar{q}^{\frac{1}{4}P_R^2} \right] \Phi(\tau)$$

Modular form of weight (λ+*d*+*k*/2 ,0) provided that ρ(*x*,*y*) satisfies:

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - 2\pi\left(x\cdot\frac{\partial}{\partial x} - y\cdot\frac{\partial}{\partial y} - \lambda - d\right)\right] \rho(x,y) = 0$$

and that $\rho(x,y)\, e^{-\pi(x^2+y^2)}$ decays sufficiently fast at infinity

The integrand is then modular invariant with: $-w = \lambda + d + \frac{k}{2}$

$$\int_{\mathcal{F}} d\mu\; \tau_2^{-\lambda/2} \sum_{P_L,P_R} \rho(P_L\sqrt{\tau_2}, P_R\sqrt{\tau_2})\; q^{\frac{1}{4}P_L^2}\, \bar{q}^{\frac{1}{4}P_R^2}\; \mathcal{F}(s,\kappa,w)$$

$$= (4\pi\kappa)^{1+\lambda/2} \int_0^\infty dt\; t^{2+\frac{2d+k}{4}-2}\, {}_1F_1\left(s - \frac{2\lambda+2d+k}{4}; 2s; t\right)\; \rho\left(P_L\sqrt{\frac{t}{4\pi\kappa}}, P_R\sqrt{\frac{t}{4\pi\kappa}}\right) \sum_{BPS} e^{-tP_L^2/4\kappa}$$

Further simplifications possible, when ρ(*x*,*y*) is polynomial

An example from non-compact heterotic vacua

Non-trivial integrals without moduli dependence

Appears in certain heterotic constructions on ALE spaces in the presence of NS5 brane backgrounds

$$\Gamma = \int_F d\mu \left(\sqrt{\tau_2}\,\eta\,\bar{\eta}\right)^3 \frac{\hat{E}_2^2\,E_8 - 2\,\hat{E}_2\,E_{10}}{\Delta} = ?$$

L. Carlevaro, E. Dudas, D. Israël
to appear

Unfold à la Niebur:

$$\frac{\hat{E}_2^2 E_4^2}{\Delta} - 2\frac{\hat{E}_2^2 E_4 E_6}{\Delta} = \tfrac{1}{5}\mathcal{F}(3,1,0) - 6\mathcal{F}(2,1,0) + 23j + 984$$

An example from non-compact heterotic vacua

$$\Gamma = \int_F d\mu \left(\sqrt{\tau_2}\,\eta\,\bar{\eta}\right)^3 \frac{\hat{E}_2^2\, E_8 - 2\,\hat{E}_2\, E_{10}}{\Delta} = -20\sqrt{2}$$

?

Modular Integrals: Current Status

1. $$\int_{\mathcal{F}} d\mu \; \Phi(\tau)$$ Stokes theorem

2. $$\int_{\mathcal{F}} d\mu \; \Gamma_{d,d}(G,B;\tau,\bar{\tau})$$ Rankin-Selberg-Zagier method

3. $$\int_{\mathcal{F}} d\mu \; \Gamma_{d+k,d}(G,B,Y;\tau,\bar{\tau})\,\Phi(\tau)$$ Unfold the elliptic genus (Niebur-Poincaré)

4. $$\int_{\mathcal{F}} d\mu \; \mathcal{Z}(\tau,\bar{\tau})$$ No general approach... yet !

Type II thermal vacua

Consider thermal (4,0) theories compactified to 2d

Right-moving SUSYs broken spontaneously (e.g. Scherk-Schwarz)

Kounnas, Porrati
Ferrara, Kounnas, Porrati
Ferrara, Kounnas, Porrati, Zwirner
Kounnas, Rostant

Class of thermal vacua where the free energy is stabilized via gravitomagnetic fluxes ("Hagedorn free")

C. Angelantonj, C. Kounnas, H. Partouche, N. Toumbas 2009
I.F., C. Kounnas, N. Toumbas, 2010

$$\int_{\mathcal{F}} d\mu\ \frac{1}{2}\sum_{a,b}(-)^{a+b}\frac{\theta^4[^a_b]}{\eta^{12}}\left(R\sum_{m,n}e^{-\frac{\pi R^2}{\tau_2}|m+\tau n|^2}(-)^{mn+ma+nb}\right)\ Z_{(8,8+4)}$$

$$\int_{\mathcal{F}_0[2]} d\mu\ \frac{\theta_2^4}{\eta^{12}}\,\Gamma_{(1,1)}[^0_1](R)\ Z_{(8,8+4)}$$

weight w = -4 modular form of $\Gamma_0(2)$

$$\Gamma_0(2)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2;\mathbb{Z})\ |\ c=0 \bmod 2\right\}$$

Express as N-P series and unfold, as before...

BUT run into convergence issues due to right-moving oscillators

Special Type II thermal vacua

However, a rare exception exists !

Massive Spectral boson-fermion Degeneracy Symmetry (MSDS)

$$\frac{1}{2}\sum_{\bar{a},\bar{b}}(-)^{\bar{a}+\bar{b}}\,\frac{\bar{\theta}^{12}[^{\bar{a}}_{\bar{b}}]}{\bar{\eta}^{12}} = \bar{V}_{24} - \bar{S}_{24} = 24$$

C. Kounnas 2008
I.F., C. Kounnas 2009

or "massive supersymmetry"

Exotic fermionic constructions in 2d where massive towers of bosonic and fermionic states are degenerate !

Special Type II thermal vacua

At the "MSDS" point

$$Z_{(8,8+4)} = E_4(\bar{V}_{24} - \bar{S}_{24}) = 24\ E_4$$

$$Z = \frac{24}{8\pi} \int_{\mathcal{F}_0[2]} d\mu \frac{E_4\, \theta_2^4}{\eta^{12}} \Gamma_{(1,1)}\left[{}^0_1\right](R)$$

weight $w = 0$ modular form of $\Gamma_0(2)$

$$\frac{E_4\, \theta_2^4}{\eta^{12}} = 16 + \frac{4096}{J_2 - 24} = \hat{\mathcal{F}}_2(1,1,0)$$

$$\frac{24}{8\pi} \int_{\mathcal{F}_0[2]} d\mu\ \mathcal{F}_2(1,1,0)\ \Gamma_{(1,1)}\left[{}^1_0\right](R; 2\tau; 2\bar{\tau})$$

$$24 \times \frac{2\sqrt{2}}{8\pi} \int_0^\infty d\tau_2\ \tau_2^{-3/2}\ e^{-\pi\tau_2\left[\frac{1}{(\sqrt{2}R)^2} + (\sqrt{2}R)^2\right]}\ \left(e^{2\pi\tau_2} - e^{-2\pi\tau_2}\right)$$

$$Z = 24 \times \left(R + \frac{1}{2R} - \left| R - \frac{1}{2R} \right| \right)$$

IF, Kounnas, Toumbas 2010
IF, Kounnas, Toumbas, Partouche 2010

Conclusions & Outlook

- ☑ Unfolding against the lattice obscures the manifest T-duality symmetries of string amplitudes

- ☑ Any weak almost holomorphic modular form can be represented as a linear combination of absolutely convergent Niebur-Poincaré series

- ☑ One-loop string amplitudes can then be represented as constrained sums over BPS states which are manifestly invariant under the T-duality group

- ☑ The singularity structure of the amplitudes becomes visible in this representation

- ☑ Results are chamber independent

- ☑ Non-trivial Wilson lines

- ☑ Insertions of lattice momenta

- ☑ Even in the absence of the lattice itself !

Conclusions & Outlook

- ☑ Generalization for modular forms of congruence subgroups of SL(2;Z) (freely-acting orbifolds)
- ☑ Higher genus amplitudes (*g*=2,3)
- ☑ Effective potential of strings at finite temperature (String Cosmology)

Happy Birthday, Costas !