September 29, 2012

Gaussian integrals

Happy birthday Costas

Of course only non-Gaussian problems are of interest!

- Matrix models of 2D-gravity

$$Z = \int dM e^{-N \mathrm{Tr} V(M)}$$

in which $M = M^\dagger$ is an $N \times N$ matrix and $V(M) = \sum t_n M^n$

- Kontsevich "Airy" matrix model

$$Z = \int dM \ \exp\left(-\frac{1}{2}\mathrm{Tr}\Lambda M^2 + \frac{i}{6}M^3\right)$$

who proved that $F = \log Z$ satisfies Witten's conjectures.

Namely define

$$t_n(\Lambda) = -(2n-1)!!\mathrm{Tr}\Lambda^{-(2n-1)}$$

then

$$F(t_0, \cdots, t_n, \cdots) = \sum \langle \tau_0^{k_0} \cdots \tau_n^{k_n} \cdots \rangle \prod_0^\infty \frac{t_n^{k_n}}{k_n!}$$

in which the coefficients $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ are the "intersection numbers of the moduli space of curves" on a Riemann surface $\mathcal{M}_{g,n}$, space of inequivalent complex structures on a Riemann surface of genus g with n marked points with $3g = n - 3 + \sum_1^n d_i$.
It follows that F satisfies the KdV hierarchy : $U = \partial^2 F / \partial t_0^2$

$$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3}$$

- Penner model

$$Z = \int dM \exp - Nt \mathrm{Tr}(\log(1 - M) + M)$$

$F = \log Z$ is the generating function of the Euler character of $\mathcal{M}_{g,n}$

$$F = \sum \chi_{g,n} N^{2-2g} t^{2-2g-n}$$

from which he obtained

$$\chi_{g,n} = (-1)^n \frac{(n + 2g - 3)!(2g - 1)}{n!(2g)!} B_{2g}$$

B_{2g} is a Bernoulli number.

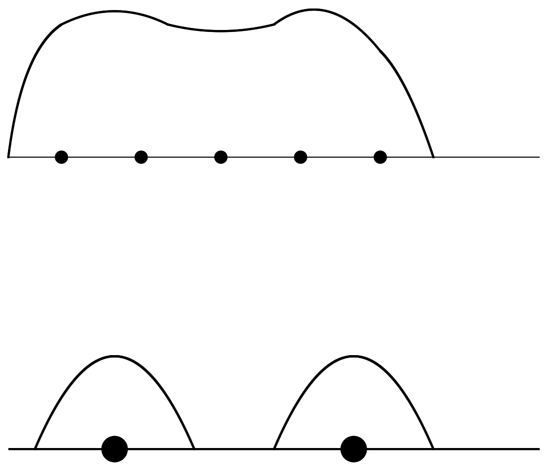
SO WHY BE INTERESTED IN GAUSSIAN INTEGRALS?

Consider Gaussian matrix models with an external matrix source

$$P_A(M) = \frac{1}{Z_A} e^{-\frac{N}{2}\mathrm{tr} M^2 - N \mathrm{tr} MA}$$

$$A = A^\dagger$$

a given matrix with eigenvalues $a_1, \cdots a_N$.
One can tune A to various situations.



Density of states

(the dots are the eigenvalues of the source)

Two basic ingredients

1. Explicit results for K-point functions (at finite N).

2. An N-K duality which exchanges the size N of the matrix, with K the number of operators.

Joint work with Shinobu Hikami

1. Correlation functions in an external source:

$$P_A(M) = \frac{1}{Z_A} e^{-\frac{N}{2}\mathrm{tr}M^2 - N\mathrm{tr}MA}$$

$$U_A(s_1, \cdots, s_K) = \frac{1}{N^K} \langle \mathrm{tr} e^{\mathrm{N s_1 M}} \cdots \mathrm{tr} e^{\mathrm{N s_K M}} \rangle$$

is a function of the eigenvalues a_α of the Hermitian source matrix A.

One point function

Using HarishChandra, Itzyson-Zuber integral over the unitary group

$$\int dU e^{\mathrm{Tr} UAU^\dagger B} = \frac{\det e^{a_i b_j}}{\Delta(A)\Delta(B)}$$

in which $\Delta(A) = \prod(a_i - a_j)$, one finds

$$U_A(s) = \frac{1}{N}\langle \mathrm{tre}^{\mathrm{NsM}} \rangle = \frac{e^{\frac{Ns^2}{2}}}{Ns} \oint \frac{du}{2i\pi} \mathrm{e}^{\mathrm{Nsu}} \prod_1^{\mathrm{N}} \left(\frac{u - a_\alpha + s}{u - a_\alpha}\right)$$

For instance for $A = 0$

$$U_0(s) = \frac{e^{\frac{Ns^2}{2}}}{Ns} \oint \frac{du}{2i\pi} e^{Nsu} (1 + \frac{s}{u})^N$$

K-point functions

$$U_A(s_1,\cdots,s_K) = \langle \mathrm{tr} e^{\mathrm{N s_1 M}} \cdots \mathrm{tr} e^{\mathrm{N s_k M}} \rangle$$

$$= e^{\sum_1^K \frac{N s_i^2}{2}} \oint \prod_{i=1}^{K} \frac{du_i e^{N s_i u_i}}{2i\pi} \mathrm{det} \frac{1}{u_i - u_j + s_i}$$

$$\times \prod_{i=1}^{K} \prod_{\alpha=1}^{N} (1 + \frac{s_i}{u_i - a_\alpha})$$

2. Duality :

We consider the average of products of K characteristic polynomials $\det(\lambda - M)$.

$$\mathrm{Tr}\frac{1}{\lambda - M} = \frac{\partial}{\partial\lambda}\frac{\det(\lambda - M)}{\det(\mu - M)}\Big|_{\mu=\lambda}$$

The K-point function is defined as

$$F_K(\lambda_1, ..., \lambda_K) = \frac{1}{Z_N} < \prod_{\alpha=1}^{K} \det(\lambda_\alpha - M) >_{A,M}$$

$$= \frac{1}{Z_N}\int dM \prod_{\alpha=1}^{K} \det(\lambda_i \cdot \mathrm{I} - \mathrm{M}) e^{-\frac{1}{2}\mathrm{Tr}(\mathrm{M}-\mathrm{A})^2}$$

Define the $K \times K$ diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_K).$$

The duality reads

$$\frac{1}{Z_N}\int [dM]_N \prod_{\alpha=1}^{K} \mathrm{det}(\lambda_\alpha \cdot \mathrm{I} - \mathrm{M}) e^{-\frac{1}{2}\mathrm{Tr}(\mathrm{M}-\mathrm{iA})^2}$$

$=$

$$(-i)^{Nk}\frac{1}{Z_k}\int [dB]_K \prod_{j=1}^{N} \mathrm{det}(a_j \cdot \mathrm{I} - B)\ e^{-\frac{1}{2}\mathrm{tr}(\mathrm{B}+\mathrm{i}\Lambda)^2}$$

where B is a $K \times K$ Hermitian matrix.

The K-point function with $N \times N$ Gaussian random matrices (in a source) is equal to an N-point function with $K \times K$ Gaussian matrices in a different source (like "color" exchanged with "flavor").

The proof is simple

$$\det(\lambda_{\alpha-M}) = \int d\bar{\theta^i}_\alpha d\theta^i{}_\alpha e^{\bar{\theta^i}_\alpha(\lambda_\alpha\delta_{ij}-M_{ij})\theta^j{}_\alpha}$$

$i = 1 \cdots N$ and repeat this K times : $\prod_{\alpha=1}^K \det(\lambda_{\alpha-M})$. Then

1. The Gaussian integral over M gives the exponential of a quartic form in the Grassmann variables :

$$\sum_{\alpha,\beta}\sum_{ij}\bar{\theta^i}_\alpha\theta^j{}_\alpha\bar{\theta^j}_\beta\theta^i{}_\beta = -\sum_{\alpha,\beta}\sum_i\bar{\theta^i}_\alpha\theta^i{}_\beta\sum_j\bar{\theta^j}_\beta\theta^j{}_\alpha$$

2. Disentangle the quartic by a Gaussian integration over a $K\times K$ matrix B

$$\int dBe^{-[\mathrm{Tr}B^2+\sum_{\alpha,\beta}B_{\alpha,\beta}\sum_i\bar{\theta^i}_\alpha\theta^i{}_\beta]}$$

with $B = B^\dagger$ is $K \times K$.

3. integrate out the Grassmannian variables $\bar{\theta}^i, \theta^i$.

$$\int d\bar{\theta^i}_\alpha d\theta^i{}_\alpha e^{\bar{\theta^i}_\alpha(\lambda_\alpha\delta_{\alpha,\beta}-B_{\alpha,\beta})\theta^i{}_\beta} = [\det_{K\times K}(\lambda_\alpha\delta_{\alpha,\beta} - B_{\alpha,\beta})]^N$$

Simplest application : recovering Kontsevich model

Consider the correlation function of the characteristic polynomials $\det(\lambda - M)$ and take a trivial unit matrix for the source $A = 1$

$$P_A(M) = \frac{1}{Z} \exp -\frac{N}{2} \mathrm{Tr}(M-1)^2$$

a trivial shift which brings one edge of Wigner's semi-circle at the origin.

$$\frac{1}{Z_N}\int [dM]_N \prod_{\alpha=1}^{K} \det(\lambda_\alpha \cdot \mathrm{I} - \mathrm{M}) e^{-\frac{N}{2}\mathrm{Tr}(\mathrm{M}-1)^2}$$

$$=$$

$$(-i)^{Nk}\frac{1}{Z_k}\int [dB]_K [\det(1-iB)]^N \; e^{-\frac{N}{2}\mathrm{tr}(\mathrm{B}+\mathrm{i}\Lambda)^2}$$

$$\det(1-iB)^N = e^{N\mathrm{tr}[-\mathrm{i}\mathrm{B}+\frac{1}{2}\mathrm{B}^2+\frac{i}{3}\mathrm{B}^3+\cdots]}$$

The linear term in B shifts Λ, the quadratic term cancels.

In a scale in which the initials λ_k are close to one, or more precisely $N^{2/3}(\lambda_k - 1)$ is finite, the large N asymptotics is given by $K \times K$ matrices B of order $N^{-1/3}$. Higher terms are negligible and we are left with terms linear and cubic in the exponent, namely

$$F_k(\lambda_1, ..., \lambda_k) \simeq e^{\frac{N}{2}\mathrm{tr}\Lambda^2} \int dB e^{i\frac{N}{3}\mathrm{tr}B^3 + iN\mathrm{tr}B(\Lambda - 1)}.$$

The original Kontsevich partition function was defined as

$$Z = \int dB e^{-\mathrm{tr}\Lambda \mathrm{B}^2 + \frac{i}{3}\mathrm{tr}\mathrm{B}^3}$$

The shift $B \to B + i\Lambda$, eliminates the B^2 term and one recovers up to a trivial rescaling the same integral.

The fact that Kontsevich's model is dual of a Gaussian model makes it easy to compute the intersection numbers.

Duality and replicas

If Λ is a multiple of the identity $\Lambda = \lambda \times 1$, the coefficients of $1/\lambda$ in the expansion of $\log Z$ are proportional to the intersection numbers of the moduli of curves with one marked point on a Riemann surface of genus g. To recover these numbers one can use replicas, i.e. the simple relation

$$\lim_{n \to 0} \frac{1}{n} \frac{\partial}{\partial \lambda} [\det(\lambda - M)]^n = \mathrm{tr} \frac{1}{\lambda - M}.$$

and

$$< [\det(\lambda - M)]^n >_{A,M} = < [\det(1 - iX)]^N >_{\Lambda, X}$$

where X is an $n \times n$ random Hermitian matrix.

Similarly with two marked points

$$U(s_1, s_2) = \langle \mathrm{Tr} e^{s_1 M} \mathrm{Tr} e^{s_2 M} \rangle$$

$$= \lim_{n_1, n_2 \to 0} \int d\lambda_1 d\lambda_2 e^{s_1 \lambda_1 + s_2 \lambda_2} \frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} \langle [\det(1 - iX)]^N \rangle_\Lambda$$

with $\Lambda = (\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2)$ degenerate n_1 and n_2 times.

Dealing with 0×0 matrices

The Gaussian average over the matrix X is given by

$$U(s_1, \cdots, s_k) = \frac{1}{n} \langle \mathrm{tr} e^{s_1 X} \cdots \mathrm{tr} e^{s_k X} \rangle$$

$$= (-1)^{k(k-1)/2} e^{\sum_1^k \frac{s_i^2}{2}} \oint \prod_1^k \frac{du_i}{2i\pi} e^{\sum_1^k (u_i s_i)}$$

$$\times \prod_1^k (1 + \frac{s_i}{u_i})^n \det \frac{1}{u_i + s_i - u_j}$$

The continuation to $n \to 0$ is straightforward

$$\lim_{n\to 0} U(s_1,\cdots,s_k) = (-1)^{k(k-1)/2} e^{\sum_1^k \frac{s_i^2}{2\lambda}}$$

$$\times \oint \prod_1^k \frac{du_i}{2i\pi} e^{\sum_1^k (u_i s_i/\lambda)} \sum_1^k \log\,(1+\frac{s_i}{u_i})\,\det \frac{1}{u_i+s_i-u_j}$$

The calculation of the contour integrals is more cumbersome, but all the integration can be done explicitly to the end and give

$$\lim_{n\to 0} U(s_1, \cdots, s_k) = \frac{\lambda}{\sigma^2} \prod_1^k 2 \sinh \frac{\sigma s_i}{2\lambda},$$

with $\sigma = s_1 + \cdots + s_k$.

This function is the generating function for the $n = 0$ limit of

$$\lim_{n\to 0} \frac{1}{n} \langle \mathrm{tr} X^{p_1} \cdots \mathrm{tr} X^{p_k} \rangle$$

.

For instance it gives

$$\lim_{n\to 0}\frac{1}{n} < (\mathrm{tr}\mathsf{X}^3)^{4\mathrm{g}-2} >= 3^{3\mathrm{g}-2}2^{-2\mathrm{g}}(6\mathrm{g}-4)!\begin{pmatrix} 4g-2 \\ g \end{pmatrix}$$

This gives the intersection numbers of the moduli of curves with one marked point.

$$< \tau_{3g-2} >_g = \frac{1}{(24)^g g!} \ \ (g = 0, 1, 2, ...).$$

Generalizations : "spin curves"

It is clear that can obtain results for curves with several marked points by the same technique (although it is difficult to go beyong low genera).

One can also "tune" the external source to obtain a generalized Kontsevich model. For instance if the source matrix A has $N/2$ eigenvalues equal to $+1$, and $N/2$ equal to -1 in the large N appropriate scaling limit

$$< \prod_{i=1}^{N} \mathrm{det}(a_i - iB) >=< [\mathrm{det}(1 + B^2)]^{\frac{N}{2}} >$$

$$= \int dB e^{-\frac{N}{4}\mathrm{tr}B^4 - iN\mathrm{tr}B\Lambda}$$

This case is nothing but the critical gap closing situation.

Again an appropriate tuning of the source matrix can yield the p-th generalization of Kontsevich's Airy matrix-model, a model introduced by Marshakov-Mironov-Morozov, defined as

$$Z = \frac{1}{Z_0} \int dB \mathrm{exp}[\frac{1}{p+1} \mathrm{tr}(\mathrm{B}^{\mathrm{p}+1} - \Lambda^{\mathrm{p}+1}) - \mathrm{tr}(\mathrm{B} - \Lambda)\Lambda^{\mathrm{p}}]$$

It can be used to recover the (p, q) models coupled to gravity.

The 'free energy' is the generating function of the generalized intersection numbers $\langle \prod \tau_{m,j} \rangle$ for moduli of curves with 'spin' j

$$F = \sum_{d_{m,j}} < \prod_{m,j} \tau_{m,j}^{d_{m,j}} > \prod_{m,j} \frac{t_{m,j}^{d_{m,j}}}{d_{m,j}!}$$

where

$$t_{m,j} = (-p)^{\frac{j-p-m(p+2)}{2(p+1)}} \prod_{l=0}^{m-1} (lp + j + 1) \mathrm{tr} \frac{1}{\Lambda^{mp+j+1}}$$

For instance for the quartic generalized model (p=3)

$$< \tau_{\frac{8g-5-j}{3},j} >_g = \frac{1}{(12)^g g!} \frac{\Gamma(\frac{g+1}{3})}{\Gamma(\frac{2-j}{3})}$$

where $j = 0$ for $g = 3m + 1$ and $j = 1$ for $g = 3m$. (For $g = 3m + 2$, the intersection numbers are zero).
Witten conjecture : the free energy which generates intersection numbers for spin curves satisfies a Gelfand-Dikii hierarchy and indeed this is true for the correlators $U(s_1, \cdots, s_K)$ providing thereby an alternative definition.

Generalization : higher Kontsevich-Penner models

$$Z = \int dM e^{Tr[M^{p+1} + k \log M + \Lambda M]}$$

- p=2 Airy
- p=-1 Penner
- p=-2 unitary matrix model

Conclusion

Topological matrix models such as generalized Kontsevich and Penner models,describe special noncritical string theories. (Maybe not surprising since integration over $\mathcal{M}_{g,n}$ is what one does in perturbative string theories.) It is still surprizing (to me) that they may be generated by simple Gaussian models ... in appropriate external sources.