# On semiclassical approximation to correlators of closed string vertex operators in $AdS_5 \times S^5$

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- R. Roiban and A.T., On semiclassical computation of 3-pt functions of closed string vertex operators in $AdS_5 \times S^5$ 1008.4921
- E. Buchbinder and A.T., On semiclassical approximation for correlators of closed string vertex operators in AdS/CFT. 1005.4516

Closely related papers:

- R. Janik, P. Surowka and A. Wereszczynski, On correlation functions of operators dual to classical spinning string states, 1002.4613
- K. Zarembo, Holographic three-point functions of semiclassical states, 1008.1059
- M.S. Costa, R. Monteiro, J.E. Santos and D. Zoakos, "On 3-point correlation functions in the gauge/gravity duality," 1008.1070

## planar N=4 SYM = tree-level $AdS_5 \times S^5$ string

solve 4d CFT = solve string theory $\subset$ 2d CFT
(i) spectrum of dimensions of primary operators
(ii) 3-point correlators
spectrum is determined by integrability (Kazakov's talk)
3-point functions not (?) controlled by integrability
(cf. flat-space string theory)

what is known about 3-point functions?

- 1/2 BPS operators = massless $AdS_5 \times S^5$ string modes
(IIB supergravity fields)
3-point functions (like dimensions) are protected
$C_{J_1 J_2 J_3} = \frac{1}{N}\sqrt{J_1 J_2 J_3}$
[Lee, Minwalla, Rangamani, Seiberg 98, ....]
general arguments for protection ...
- correlators involving non-BPS operators (massive string modes)

weak coupling :
direct 1-loop computations
$C_{123} = C^{(0)}_{123}(1 + \lambda \sum_{n=1}^{3} c_n \gamma_n^{(1)} + ...)$
$\Delta_n = \Delta_n(0) + \gamma_n \ , \qquad \gamma_n = \lambda \gamma_n^{(1)} + \lambda^2 \gamma_n^{(2)} + ...$
[Bianchi, Kovacs, Rossi, Stanev 01;
Okuyuama, Tseng 04; Grosardt, Plefka 10]
may also use perturbed spin chain Hamiltonian to
compute some correlation functions [Roiban, Volovich 04]

strong coupling: little is known
[some earlier studies in near-BMN limit are inconclusive]
if, e.g., conjecture exponentiation ($\exp\sum_{n=1}^{3} c_n \gamma_n$) then
$e^{a\sqrt{\lambda}}$ behaviour at strong coupling for operators
with large quantum numbers $\Delta \sim S \sim \sqrt{\lambda}$
can be captured by semiclassical approximation
as for 2-point functions ?
[Janik, Surowka, Wereszczynski 10; Buchbinder, AT 10]

special case:
two "heavy" operators with $\Delta_{1,2} \sim \sqrt{\lambda}$
one "light" operator $\Delta_3 \ll \sqrt{\lambda}$:
correlator saturated by semiclassical trajectory
determined by "heavy" states

• "light" state is BPS:
K. Zarembo, 1008.1059;
M.S. Costa, R. Monteiro, J.E. Santos, D. Zoakos, 1008.1070
• "light" state is non-BPS:
R. Roiban, A.T., 1008.4921

Lessons ?
Extension of similar semiclassical approach to
special 4-point functions with non-BPS states ?

Planar N=4 SYM – $AdS_5 \times S^5$ string duality:
4d CFT vs 2d CFT
planar correlators of single-tr conformal primary ops in SYM
= correlators of closed-string vertex ops on 2-sphere
equality of generating functionals

$$\langle e^{\Phi \cdot O} \rangle_{4d} = \langle e^{\Phi \cdot V} \rangle_{2d}$$

$O$ = primary SYM operator of dimension $\Delta$
$V$ = corresponding marginal string vertex operator

$$\Phi \cdot O = \int d^4x' \; \Phi(x') O(x')$$

$$\Phi \cdot V = \int d^4x' \; \Phi(x') V(x', z, ...)$$

$$V = \int d^2\xi \; \mathrm{V}(\xi; x', z, ...)$$

Poincare patch: $ds^2 = z^{-2}(dz^2 + dx^m dx_m)$
symbolic structure of vertex operators

$$\mathrm{V} = K(\partial X \partial X + ...)$$

$$K(x - x'; z) = c\big[z + z^{-1}(x - x')^2\big]^{-\Delta}$$

$$K(x - x'; z)_{z \to 0} = \delta^{(4)}(x - x')$$

2-point and 3-point correlators special:
$x$-dependence fixed by 4d conf. invariance

$$\langle V_1(\mathrm{x}) V_2(\mathrm{x}')\rangle_{4d} = \frac{\delta_{\Delta_1,\Delta_2}}{|\mathrm{x}-\mathrm{x}'|^{2\Delta_1}}$$

$$\langle V_1(\mathrm{x}) V_2(\mathrm{x}') V_3(\mathrm{x}'')\rangle_{4d}$$

$$= \frac{C_{123}}{|\mathrm{x}-\mathrm{x}'|^{\Delta_1+\Delta_2-\Delta_3}|\mathrm{x}-\mathrm{x}''|^{\Delta_1+\Delta_3-\Delta_2}|\mathrm{x}'-\mathrm{x}''|^{\Delta_2+\Delta_3-\Delta_1}}$$

Similar relations for correlators of $O$'s corresponding $V$'s

Problems :

- compute the spectrum, i.e. functions $\Delta(\lambda, Q)$

$\lambda = g^2_{\rm YM} N$, string tension $T = \frac{\sqrt{\lambda}}{2\pi}$
$Q = (S_1, S_2, J_1, J_2, J_3; ..., ...)$ – charges characterizing $O_\Delta$

- compute $C_{123}(\lambda, Q_1, Q_2, Q_3)$

higher-point correlators – via OPE

## General idea of semiclassical approach:

$V \sim (...)^{\Delta}(...)^{Q}$
so if $\Delta$ and the charges $Q$ scale as $T = \frac{\sqrt{\lambda}}{2\pi}$
they produce terms in

$$\langle V...V \rangle = \int [dx]\, V...V\ \exp(-T \int d^2\xi\, \partial X \partial X + ...)$$

of same order as string action –
$\sqrt{\lambda} \gg 1$ limit is dominated by classical trajectory
with vertex operators providing "source terms"
$\to$ may lead to a prediction for strong-coupling behaviour
of corresponding gauge theory correlators

similar idea for 2-point functions [Polyakov 02; A.T. 03]
and correlators with Wilson loops [Zarembo 02; Tsuji 06]

Consider

$$\mathrm{K}_{n,m} = \langle V_{H_1}(\mathrm{x}_1)...V_{H_n}(\mathrm{x}_n)\ V_{L_1}(\mathrm{x}_{n+1})...V_{L_m}(\mathrm{x}_{n+m})\rangle$$

$V_H$ – “heavy” (“semiclassical”) with $\Delta_H \sim Q \sim \sqrt{\lambda} \gg 1$
$V_L$ – “light” (or “quantum”) with $Q \sim 1$
and $\Delta_L \sim \sqrt[4]{\lambda}$ for massive string states
or $\Delta_L \sim 1$ for “massless” (BPS) string states
may expect that for large $\sqrt{\lambda}$ leading contribution
given by semiclassical string trajectory determined
by the “heavy” operator insertions

Strategy:
(i) construct classical solution that determines
large $\sqrt{\lambda}$ contribution to $\mathrm{K}_n = \langle V_{H_1}(\mathrm{x}_1)...V_{H_n}(\mathrm{x}_n)\rangle$
(ii) compute $\mathrm{K}_{n,m}$ by evaluating
$V_{L_1}(\mathrm{x}_{n+1})...V_{L_m}(\mathrm{x}_{n+m})$ on that classical solution

motivation: contribution of “source” terms from
“light” operators are subleading at $\sqrt{\lambda} \gg 1$

3-point functions:
semiclassical trajectory controlling $\lambda \gg 1$ limit of
$\langle V_{H_1}(\mathrm{x}_1) V_{H_2}(\mathrm{x}_2) V_{H_3}(\mathrm{x}_3)\rangle$
not known [cf. Janik et al, 10] (talk by Romuald Janik)
but can use semiclassical trajectory for $\langle V_{H_1}(\mathrm{x}_1) V_{H_2}(\mathrm{x}_2)\rangle$
[A.T. 03, Buchbinder, A.T. 10]
to compute leading contribution to
$\langle V_{H_1}(\mathrm{x}_1) V_{H_2}(\mathrm{x}_2) V_L(\mathrm{x}_3)\rangle$

Examples with $V_H$ corresponding to some
semiclassical string states with large spin in $S^5$ and
- $V_L$ as chiral primary scalar BPS state [Zarembo, 10]
- $V_L$ as dilaton [Costa et al 10] (talk by Miguel Costa)

Our work [Roiban, A.T. 10]:
(i) more general choices of $V_H$:
twist operators or "small" strings dual to "short" operators
(ii) cases when $V_L$ represents massive string modes

# Examples of string vertex operators

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\xi \, \Big( \partial Y_M \bar{\partial} Y^M + \partial X_k \bar{\partial} X_k + \text{fermions} \Big)$$

$Y_M Y^M = -Y_0^2 - Y_5^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1$
$X_k X_k = X_1^2 + ... + X_6^2 = 1$
$V = V(Y, X, \psi)$ – highest weight states of $SO(2,4) \times SO(6)$
particular linear combinations of products of $Y_M, X_k$
and derivatives that are dim 2 eigenvectors of 2d anom dim op
leading $\sqrt{\lambda} \gg 1$: ignore fermions and $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms in $V$'s
global and Poincaré coordinates in $AdS_5$

$$Y_5 + iY_0 = \cosh\rho \, e^{it} \,, \qquad Y_1 + iY_2 = \sinh\rho \, \cos\theta \, e^{i\phi_1} \,,$$
$$Y_3 + iY_4 = \sinh\rho \, \sin\theta \, e^{i\phi_2} \,, \qquad Y_m = \frac{x_m}{z} \,,$$
$$Y_4 = \frac{1}{2z}(-1 + z^2 + x^m x_m) \,, \qquad Y_5 = \frac{1}{2z}(1 + z^2 + x^m x_m) \,,$$

$SO(2,4)$ rep labelled by $SO(2) \times SO(2) \times SO(2)$ Cartans $(E, S_1, S_2)$
wave function or a vertex op for state with AdS energy $E$

should contain a factor $(Y_5 + iY_0)^{-E} = (\cosh\rho)^{-E}\, e^{-iEt}$
or if labelled by $SO(1,1)$: $(Y_5 + Y_4)^{-\Delta}$
Euclidean continuation:

$$t_e = it\,, \qquad Y_{0e} = iY_0\,, \qquad x_{0e} = ix_0\,,$$

$SO(2,4) \to SO(1,5)$, $\;Y_{0e} \leftrightarrow Y_4$, $\;E \leftrightarrow \Delta$,
$(Y_5 + iY_0)^{-E} \to \mathrm{Y}_+^{-\Delta}$, $\qquad \mathrm{Y}_+ \equiv Y_5 + Y_4$

$$K(x,z) = k_\Delta\, (\mathrm{Y}_+)^{-\Delta} = k_\Delta\, \left(z + z^{-1} x^m x_m\right)^{-\Delta}$$

$K(x, z \to 0) = \delta^{(4)}(x)$
solution of scalar Laplace eq in $AdS_5$ with mass $m^2 = \Delta(\Delta-4)$
unintegrated vertex operator

$$\mathrm{V} \sim (\mathrm{Y}_+)^{-\Delta}\, (\partial^s Y)^r ... (\bar\partial^m X)^n \equiv (\mathrm{Y}_+)^{-\Delta}\, U(Y, X, ...)$$

integrated vertex op at point of bndry of euclidean Poincaré patch

$$V(\mathrm{x}) = \int d^2\xi\, \mathrm{V}\big(x(\xi) - \mathrm{x}\big) = \int d^2\xi\, [K(x(\xi) - \mathrm{x}, z(\xi))]^{-\Delta}\, U[...]$$

## Massless mode vertex operators

• **Dilaton**

$$\mathrm{V}_J = (\mathrm{Y}_+)^{-\Delta} \, (\mathrm{X}_x)^J \, \left(\partial Y_M \bar\partial Y^M + \partial X_k \bar\partial X_k + \text{fermions}\right) \, ,$$

$$\mathrm{X}_x \equiv X_1 + i X_2 = \cos\vartheta \, e^{i\varphi} \, , \qquad \Delta = 4 + J$$

dual to $\mathrm{Tr}(F_{mn}^2 Z^J + ...)$

• **Superconformal primary scalar**
$[0, J, 0], \quad J \geqslant 2$, of $SO(6)$
$\Delta = J, \quad$ dual to $\mathrm{Tr} Z^J$
originates from trace graviton in $S^5$ directions
induces components of graviton in $AdS_5$ directions
and mixes with the RR 5-form
relevant part of bosonic term in vertex op. [Berenstein et al 98]

$$\mathrm{V}_J = (\mathrm{Y}_+)^{-\Delta} \, \mathrm{X}_x^J \left[ z^{-2} (\partial x^m \bar\partial x_m - \partial z \bar\partial z) - \partial X_k \bar\partial X_k \right]$$

4+6 split: flipped sign of 6-part

## String states on leading Regge trajectory

flat space: spin $S$ state

$V_S = e^{-iEt}\big(\partial \mathrm{x}_x \bar{\partial} \mathrm{x}_x\big)^{\frac{S}{2}}, \qquad \mathrm{x}_x = x_1 + ix_2,$

$E = \sqrt{\frac{2}{\alpha'}(S-2)}$

$AdS_5 \times S^5$ analogs ($E \to \Delta$)

$$\mathrm{V}_S = (\mathrm{Y}_+)^{-\Delta}\big(\partial \mathrm{Y}_x \bar{\partial} \mathrm{Y}_x\big)^{\frac{S}{2}} + ... \qquad \mathrm{Y}_x = Y_1 + iY_2 \ ,$$

$$\mathrm{V}_J = (\mathrm{Y}_+)^{-\Delta}\big(\partial \mathrm{X}_x \bar{\partial} \mathrm{X}_x\big)^{\frac{J}{2}} + ... \qquad \mathrm{X}_x = X_1 + iX_2$$

$\mathrm{V}_J$ may mix with ($p, q = 0, ..., \frac{J}{4};\ l, k = 1, ..., 6$)

$$(\mathrm{X}_x)^{2p+2q}(\partial \mathrm{X}_x)^{\frac{J}{2}-2p}(\bar{\partial} \mathrm{X}_x)^{\frac{J}{2}-2q}(\partial X_\ell \partial X_\ell)^p(\bar{\partial} X_k \bar{\partial} X_k)^q \ ,$$

true vertex ops= eigenvectors of 2d anom dim matrix

cf. solving Lichnerowitz type eq for tensor wave function

$\widehat{\gamma}\Psi = \Big[2 - S + \frac{1}{2}\alpha' \nabla^2 + \sum c_k \alpha'^k (R....)^n ... \nabla^p\Big]\Psi = 0$

considering such ops as “heavy” ( treated semiclassically)

may ignore mixing to leading order:

need only solution they source to have definite energy or $\Delta$

## Singlet massive string states

special massive string state vertex operators
with finite quantum numbers
with leading-order bosonic part known explicitly
singlet operators that do not mix with others
to leading order in $\frac{1}{\sqrt{\lambda}}$ [A.T. 03]
can be used as "light" vertex operators

$$\mathrm{V}_r = (\mathrm{Y}_+)^{-\Delta}(\partial X_k \partial X_k \bar{\partial} X_\ell \bar{\partial} X_\ell)^{r/2} \ , \qquad r = 2, 4, \ldots$$

ignoring fermionic contributions, marginality cond

$$0 = \widehat{\gamma} = 2 - 2r + \frac{1}{2\sqrt{\lambda}}\Big[\Delta(\Delta - 4) + 8r\Big] + O(\frac{1}{(\sqrt{\lambda})^2})$$

$$\Delta = 2\sqrt{r-1}\sqrt[4]{\lambda} + 2 - \frac{2r-1}{\sqrt{r-1}\sqrt[4]{\lambda}} + \mathcal{O}(\frac{1}{(\sqrt[4]{\lambda})^3})$$

the corresponding singlet scalar field should satisfy

$$(-\nabla^2 + M^2 + \ldots)\Phi = 0, \quad M^2 = \Delta(\Delta - 4) = 4(r-1)\sqrt{\lambda} + \ldots$$

$AdS_5$ counterpart:

$$\mathrm{V}_k = (\mathrm{Y}_+)^{-\Delta}(\partial Y_M \partial Y^M \bar{\partial} Y_K \bar{\partial} Y^K)^{k/2} \ , \qquad k = 2, 4, \ldots$$

$k = 2$ represents a massive state on first excited string level
should be dual to a member of Konishi multiplet
[Bianchi, Morales, Samtleben 03; Roiban, A.T. 09]

# Semiclassical approximation for 2-point correlator

point-like string with large orbital momentum in $S^5$
$t = \kappa\tau$ (in $AdS_5$) and $\varphi = \kappa\tau$ (in $S^5$)
massive AdS geodesic, reaches bndry after Euclidean cont.

$$z = [\cosh(\kappa\tau_e)]^{-1}, \quad x_{0e} = \tanh(\kappa\tau_e)\,, \quad \varphi = -i\kappa\tau_e, \; \tau_e = i\tau$$

$$\tau_e \to \pm\infty : \quad z \to 0, \quad x_{0e} = \pm 1, \quad x_i = 0$$

vertex ops at $\tau_e = \pm\infty$ on Euclidean 2d cylinder
mapped to $\xi_1$ and $\xi_2$ on the $\xi$ complex plane by
$e^{\tau_e + i\sigma} = \frac{\xi - \xi_2}{\xi - \xi_1}$
solution with given charges on a Lorenzian 2d cylinder
mapped onto the complex plane:
stationary trajectory for 2-point function $\langle VV \rangle$ with given charges
"delta-function" sources from $V$'s at $\xi_1$ and $\xi_2$
matching onto sources relates parameters of solution
to quantum numbers ($\Delta$, $J$, ...) of $V$'s

## Example: large spin operator in AdS

$\langle V_S(\mathrm{x}_1) V_S^*(\mathrm{x}_2)\rangle$

$V_S(\mathrm{x}) = \int d^2\xi \, \left[z + z^{-1}(x-\mathrm{x})^2\right]^{-\Delta} \left[\partial \mathrm{Y}_{x-\mathrm{x}} \, \bar\partial \mathrm{Y}_{x-\mathrm{x}}\right]^{S/2}$

$\mathrm{Y}_x = Y_1 + iY_2 = \frac{x_1 + i x_2}{z}$

if $\Delta \sim S \sim \sqrt{\lambda} \gg 1, \quad \mathcal{S} = \frac{S}{\sqrt{\lambda}} \gg 1$

semicl. trajectory = conf. transformed euclidean contn

of large spin limit of spinning folded string solution

$t = \kappa\tau \,, \qquad \phi = \kappa\tau \,, \qquad \rho = \mu\sigma \,, \quad \kappa = \mu \approx \frac{1}{\pi} \ln \mathcal{S} \gg 1$

Euclidean solution in Poincaré coordinates

$z = [\cosh(\kappa\tau_e)\cosh(\mu\sigma)]^{-1} \,, \qquad x_{0e} = \tanh(\kappa\tau_e) \,,$

$x_1 = \tanh(\mu\sigma) \,, \qquad x_2 = -i \tanh(\mu\sigma) \, \tanh(\kappa\tau_e) \,,$

$x_\pm \equiv x_1 \pm i x_2 = r e^{\pm i\phi} = \frac{\tanh(\mu\sigma)}{\cosh(\kappa\tau_e)} \, e^{\pm\kappa\tau_e} \,,$

$z^2 + x_{0e}^2 + x_1^2 + x_2^2 = 1$

Lorenzian: string moves to center of AdS rotating and stretching

Euclidean continuation: gives complex world surface

approaching boundary $z = 0$

$\tau_e \to \pm\infty$: $x_{0e} \to \pm 1$ and "light-like" lines in $(x_1, x_2)$:

$$\begin{aligned} \tau_e \to +\infty \ : \quad & z \to 0\,, \quad x_{0e} \to 1\,, \\ & x_+ \to 2\tanh(\mu\sigma)\,, \quad x_- \to 0 \\ \tau_e \to -\infty \ : \quad & z \to 0\,, \quad x_{0e} \to -1\,, \\ & x_+ \to 0\,, \quad x_- \to 2\tanh(\mu\sigma) \end{aligned}$$

surface does not simply end at 2 points at boundary
but no such requirement:
trajectory should be "sourced" by 2 vertex ops at $\mathrm{x}_1$ and $\mathrm{x}_2$
boundary values of classical string coordinates need not
coincide with positions of vertex operators
above choice for $\mathrm{x}_1 = (1, 0, 0, 0)$ and $\mathrm{x}_2 = (-1, 0, 0, 0)$

similarly for large $S$, large $J = \sqrt{\lambda}\mathcal{J}$ in $S^5$

$$V_{S,J}(0) = \int d^2\xi \; (\mathrm{Y}_+)^{-\Delta} \; (\mathrm{X}_x)^J \; \left(\partial \mathrm{Y}_x \, \bar{\partial} \mathrm{Y}_x\right)^{S/2}$$

euclidean semiclassical solution

$$t_e = \kappa\tau_e \; , \quad \phi = -i\kappa\tau_e \; , \qquad \rho = \mu\sigma \; , \qquad \varphi = -i\nu\tau_e$$

$$\kappa = \sqrt{\mu^2 + \nu^2} \; , \qquad \mu \approx \tfrac{1}{\pi} \ln \mathcal{S} \gg 1 \; , \quad \nu = \mathcal{J}$$

$$E - S = \sqrt{J^2 + \tfrac{\lambda}{\pi^2} \ln^2 \mathcal{S}} = \tfrac{\sqrt{\lambda}}{\pi} \sqrt{\ell^2 + 1} \; \ln \mathcal{S} \; , \qquad \ell \equiv \tfrac{\nu}{\mu}$$

dual to $\mathrm{Tr}(D^S Z^J)$ operator in gauge theory

# 3-point functions of two "heavy" and one "light" states

leading $\sqrt{\lambda} \gg 1$ order of $\langle V_{H_1}(\mathrm{x}_1) V_{H_2}(\mathrm{x}_2) V_L(\mathrm{x}_3)\rangle$
for $\Delta_{H_1} = \Delta_{H_2} \sim \sqrt{\lambda} \gg \Delta_L \equiv \Delta$

(i) find semiclassical trajectory for $\langle V_{H_1}(\mathrm{x}_1) V_{H_2}(\mathrm{x}_2)\rangle$
(ii) evaluate $V_L(\mathrm{x}_3)$ on it
conformal invariance: sufficient to consider $\mathrm{x}_3 = (0,0,0,0)$

$$V_L(0) = \int d^2\xi \ (\mathrm{Y}_+)^{-\Delta_L} \ U[x(\xi), z(\xi), X(\xi)]$$

for all simple classical solutions for $V_H$

$$z^2 + x_m x^m = 1 \,, \quad \text{i.e.} \quad Y_4 = 0 \,, \quad Y_5 = \mathrm{Y}_+ = z^{-1}$$

and approach boundary at $|\mathrm{x}_1| = 1, \ |\mathrm{x}_2| = 1$

$$C_{123} = \frac{\langle V_{H_1}(\mathrm{x}_1) V_{H_2}(\mathrm{x}_2) V_L(0)\rangle}{\langle V_{H_1}(\mathrm{x}_1) V_{H_2}(\mathrm{x}_2)\rangle}$$
$$= c_\Delta \int d^2\xi \ z^\Delta(\xi) \ U[x(\xi), z(\xi), X(\xi)]$$

## $V_H$ corresponding to large spin $(S, J)$ string

### $V_L$ as dilaton operator

$$C_{123} = c_\Delta \int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi} d\sigma \; z^\Delta \; U \; ,$$

$$U = (\mathrm{X}_x)^j \left[ z^{-2}(\partial x_m \bar{\partial} x^m + \partial z \bar{\partial} z) + \partial X_k \bar{\partial} X_k \right]$$

$$c_\Delta = \frac{j+3}{2^{j/2+1}\pi^2} \; , \qquad \Delta = 4 + j, \quad j \ll J$$

$$C_{123} = 4 c_\Delta \int_{-\infty}^{\infty} d\tau_e \int_0^{\frac{\pi}{2}} d\sigma \; \frac{2\mu^2 \; e^{j\nu\tau_e}}{\left[\cosh(\mu\sigma)\cosh(\kappa\tau_e)\right]^\Delta}$$

$$\kappa^2 = \mu^2 + \nu^2, \quad \mu = \tfrac{1}{\pi} \ln \mathcal{S} \gg 1, \quad \nu = \mathcal{J} = \tfrac{J}{\sqrt{\lambda}}, \; \mathcal{S} = \tfrac{S}{\sqrt{\lambda}}$$

$$C_{123} \; \sim \; {}_2F_1\Big( \tfrac{1}{2}, \tfrac{1}{2}(5+j), \tfrac{3}{2}, -\sinh^2(\tfrac{\pi}{2}\mu) \Big) \; ,...$$

$j=0,\ \Delta=4$, large $S$ limit:

$$C_{123} \sim \frac{\ln S}{\sqrt{J^2+\frac{\lambda}{\pi^2}\ln^2 S}}$$

• $J \gg \frac{\sqrt{\lambda}}{\pi}\ln S$: $\quad C_{123}\to 0$
dilaton does not couple to BMN states
• $J \ll \frac{\sqrt{\lambda}}{\pi}\ln S$: $\quad C_{123}\to$ const
dilaton couples to massive states as expected
$\mathrm{S}=\int d^{10}x\,\sqrt{g}\big(\partial^\mu\Psi\partial_\mu\Psi+M^2e^{\gamma\Phi}\Psi^2+...\big)$

relation to dimension of $V_H$:

$$\sqrt{\lambda}\frac{\partial}{\partial\sqrt{\lambda}}\,\Delta_{S,J}=\frac{\ln^2 S}{\pi^2\sqrt{J^2+\frac{\lambda}{\pi^2}\ln^2 S}}+...,$$

$$\Delta_{S,J}=S+\sqrt{J^2+\frac{\lambda}{\pi^2}\ln^2 S}+...$$

cf. "soft dilaton" theorem, but difference by "IR factor $\ln S$"

## $V_L$ as superconformal primary scalar

$\Delta = j \ll J, \quad c_\Delta = \frac{(j+1)\sqrt{j}}{2^{j+2}\pi N}$

$$C_{123} = 4c_\Delta \int_{-\infty}^{\infty} d\tau_e \int_0^{\frac{\pi}{2}} d\sigma \, \frac{2\, e^{j\nu\tau_e}\left[\frac{\kappa^2}{\cosh^2(\kappa\tau_e)} - \mu^2 \tanh^2(\mu\sigma)\right]}{\left[\cosh(\mu\sigma)\cosh(\kappa\tau_e)\right]^\Delta}$$

- $\mathcal{J} \gg \ln \mathcal{S}$

formal limit $\mu \to 0$

$$C_{123} = \frac{2^{j+2}\pi c_\Delta}{j+1}\, \mu \frac{\pi \mathcal{J}}{\ln \mathcal{S}} + ... \to \frac{1}{N} J\sqrt{j}$$

agrees with result for 3 BMN states $C_{123} = \frac{1}{N}\sqrt{j_1 j_2 j_3}$
with $j_1 = j_2 = J, \; j_3 = j$

- $\ln \mathcal{S} \gg \mathcal{J}$

$$C_{123} = 4c_\Delta \pi \frac{\Gamma((j+2)/2)}{j\Gamma((j+3)/2)}\left[\frac{(j-1)\Gamma(j/2)}{\Gamma((j+1)/2)} + \frac{2^j}{j\,\mathcal{S}^{j/2}} + \ldots\right]$$

approaches const as expected

## $V_L$ as fixed-spin operator on leading Regge trajectory

$V_H$: spin $S \sim \sqrt{\lambda}$, $\Delta_S = S + .... \sim \sqrt{\lambda} \gg 1$

$V_L$: spin $s$, $\Delta_s = \sqrt{2(s-2)}\sqrt[4]{\lambda} + \ldots\,, \quad s \ll S,\ \Delta_s \ll \Delta_S$

$$U = (\partial \mathrm{Y}_x \bar{\partial} \mathrm{Y}_x)^{s/2} = e^{2s\kappa\tau_e} \Big[\mu^2 \cosh^2(\mu\sigma) + \kappa^2 \sinh^2(\mu\sigma)\Big]^{s/2}$$

$$C_{123} \sim \mu^{s-2} \sim (\ln S)^{s-2}$$

$C_{123} \sim$ (anom. dim. of heavy operator)$^{\text{string level of light operator}}$

## $V_L$ as singlet massive scalar operator

massive state at level $r-1$:
$\Delta_r = 2\sqrt{(r-1)\sqrt{\lambda}} + ... \ll \quad \Delta_S \sim \sqrt{\lambda}$
on $(S, J)$ solution

$$U = (\partial X_k \partial X_k \bar{\partial} X_\ell \bar{\partial} X_\ell)^{r/2} = (\partial Y_M \partial Y^M \bar{\partial} Y_K \bar{\partial} Y^K)^{r/2} = \mathcal{J}^{2r}$$

$$C_{123} = B(r,\ell)(\ln \mathcal{S})^{2r-2}\,(\mathcal{S}^{1/2} - \mathcal{S}^{-1/2})\;{}_2F_1\Big(\tfrac{1}{2}, \tfrac{1}{2}(\Delta_r+1), \tfrac{3}{2}, -\tfrac{1}{4}(\mathcal{S} - 2 + \mathcal{S}^{-1})\Big)$$

$$B(r,\ell) = c_\Delta \frac{2^{\Delta_r - 2}\,[\Gamma(\Delta_r/2)]^2}{\pi^{2r-2}\,\Gamma(\Delta_r)}\;\frac{\ell^{2r}}{\sqrt{1+\ell^2}}\,, \qquad\qquad \ell \equiv \frac{\pi \mathcal{J}}{\ln \mathcal{S}}$$

$\ln \mathcal{S} \gg 1, \ \mathcal{J} \gg 1$, fixed $\ \ell$

$$C_{123} \sim \frac{\ell^{2r}}{\sqrt{1+\ell^2}}\,(\ln \mathcal{S})^{2r-2} \ \sim \ \frac{J^{2r}}{\ln S\,\sqrt{J^2 + \frac{\lambda}{\pi^2}\ln^2 S}}$$

$C_{123} \sim$ (anom. dim. of heavy operator)$^{\text{string level of light operator}}$

Large $s$ behaviour:

$$C_{123} \approx \frac{c_{\Delta_s}}{\pi^{s-2}} \; e^{(s-2)\ln\ln \mathcal{S} + h_{\tau_e}(s) + h_\sigma(s)}$$

$$h_{\tau_e} = \left(\tfrac{1}{2}\Delta_s - s\right)\ln\left(1 - \frac{2s}{\Delta_s}\right) + \left(\tfrac{1}{2}\Delta_s + s\right)\ln\left(1 + \frac{2s}{\Delta_s}\right) ,$$

$$h_\sigma = \tfrac{1}{2}\Delta_s \ln 2 + \tfrac{1}{2}\Delta_s \ln\left(1 - \frac{s}{\Delta_s}\right) - s\ln\left(\frac{\Delta_s}{s} - 1\right)$$

If formally assume that $s \sim \sqrt{\lambda} \gg 1$
then $\Delta_s = \sqrt{2(s-2)}\sqrt[4]{\lambda} + ... \sim \sqrt{\lambda}$
so function in exponent $\sim \sqrt{\lambda}$
as should be in semiclassical limit
may help shed light on the case when all 3 states are “heavy” ?

## $V_H$ corresponding to "small" circular string in $S^5$

state with $J_1 = J_2 = J \neq J_3$
$\langle V_H V_H \rangle$ determined by

$$t = \kappa\tau, \quad X_{1+i2} = a\, e^{iw\tau + i\sigma}, \quad X_{3+i4} = a\, e^{iw\tau - i\sigma}, \quad X_{5+i6} = \sqrt{1-2a^2}\, e^{i\nu\tau}$$
$$w = \sqrt{1+\nu^2}, \quad \kappa = \sqrt{4a^2+\nu^2}, \quad J_1 = J_2 = J = \sqrt{\lambda}a^2 w, \quad J_3 = \sqrt{\lambda}(1-2a^2)\nu$$

euclidean trajectory:
same as for massive AdS geodesic + complex surface for $X_k$

## $V_L$ as dilaton operator

$\Delta = 4 + j$

$$C_{123} = c_\Delta 8\pi a^2 (1-2a^2)^{j/2} \int_{-\infty}^{\infty} d\tau_e \frac{e^{j\nu\tau_e}}{\left[\cosh(\kappa\tau_e)\right]^\Delta}$$

$\nu \sim J_3 = 0$:

$$C_{123} \sim \sqrt{J}\,(1 - 2\frac{J}{\sqrt{\lambda}})$$

case of $a = \frac{1}{\sqrt{2}}$: "large" circular string with $J_1 = J_2, \ J_3 = 0$
$\Delta_J = \sqrt{4J^2 + \lambda}$

$$C_{123} = \frac{16}{3}\pi c_\Delta \ \frac{\sqrt{\lambda}}{\sqrt{4J^2+\lambda}} \sim \sqrt{\lambda}\frac{\partial}{\partial\sqrt{\lambda}}\Delta_J$$

similar observation made by Costa et al

$V_L$ as singlet massive scalar

for "small" string with $\mathcal{J}_1 = \mathcal{J}_2 \equiv \mathcal{J}$, $\mathcal{J}_3 \to 0$, $\kappa = \sqrt{2\mathcal{J}}$

$$C_{123} \sim (\sqrt{J})^{2r-1} \sim (\Delta_J)^{2r-1}$$

again scales as power of level of the "light" string state

small $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ limit may be used to approximate string states
with fixed quantum number $J$
e.g., $r = 2$: first excited string level
shed light on 3-point functions involving Konishi operator ?

# Concluding remarks

• semiclassical approximation:
novel data for 3-point functions involving massive string states
• extension of semiclassical approach to 4-point functions
$\langle V_H V_H V_L V_L \rangle$ (in progress)
relevant example: all 4 states are chiral primary
[0,p,0] with arbitrary p [Uruchurtu 08]
• extensions to other states:
need to know vertex operators of $AdS_5 \times S^5$ superstring

• hidden symmetries that control 3-point coupling?
string field theory 3-vertex for $AdS_5 \times S^5$ superstring ?
• role of integrability?
relation to semiclassical approach [Alday, Maldacena, et al ]
to open string correlators / Wilson loops?