Holography for Schrödinger

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Crete, September 2010

Introduction

- Gauge/gravity dualities have become an important new tool in extracting strong coupling physics.
- The best understood examples of such dualities involve relativistic quantum field theories.
- Strongly coupled non-relativistic QFTs are common place in condensed matter physics and elsewhere.
- It is natural to wonder whether holography can be used to obtain new results about such non-relativistic strongly interacting systems.

The non-relativistic conformal group

In non-relativistic physics the Poincaré group is replaced by the Galilean group. It consists of

- the temporal translation $\mathcal{H}$, spatial translations $\mathcal{P}^i$, rotations $\mathcal{M}^{ij}$, Galilean boosts $\mathcal{K}^i$ and the mass operator $\mathcal{M}$.

The conformal extension adds to these generators

- the non-relativistic scaling operator $\mathcal{D}$ and the non-relativistic special conformal generator $\mathcal{C}$.

The scaling symmetry acts as

$$t \to \lambda^2 t, \qquad x^i \to \lambda x^i$$

Schrödinger group

- This is the maximal kinematical symmetry group of the free Schrödinger equation [Niederer (1972)], hence its name: Schrödinger group $Sch(d)$.

Interacting systems that realize this symmetry include:

- Non-relativistic particles interacting through an $1/r^2$ potential.
- Fermions at unitarity. (Fermions in three spatial dimensions with interactions fine-tuned so that the s-wave scattering saturates the unitarity bound). This system has been realized in the lab using trapped cold atoms [O'Hara et al (2002) ...] and has created enormous interest.

Holography for Schrödinger

Motivated by such applications [Son (2008)] and [K. Balasubramanian, McGreevy (2008)] considered

$$ds^2 = -\frac{b^2 du^2}{r^{2z}} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

- When $b = 0$ this is the AdS_{d+1} metric.
- For $z = 2$ this metric realizes geometrically the Schrödinger group in $(d-1)$ dimensions.
- In order for the mass operator $\mathcal{M}$ to have discrete eigenvalue lightcone coordinate v must be compactified.

Bulk system

This metric solves the field equations for e.g.

- gravity coupled to **massive vectors**
- **topologically massive gravity (TMG)** with $\mu = 3$

In the latter case the solution is called "null warped AdS_3" and it was conjectured to be dual to a 2d CFT with certain (c_L, c_R) [Anninos et al (2008)].

$\rightarrow$ This is a rather different proposal for the physics of the solution.

References

- M. Guica, K. Skenderis, M. Taylor, B. van Rees
  Holography for Schrödinger backgrounds,
  1008.1991
- K. Skenderis, M. Taylor, B. van Rees
  Asymptotically Schrödinger
- R. Caldeira-Costa and M. Taylor
  Holography for chiral scale-invariant models
  ($z \neq 2$ case)

plus other work in progress...

The issues

- These spacetimes are not asymptotically AdS and so the usual holographic set up is not automatically applicable.

Even basic issues such as

- is the dual theory a local QFT?
- what is the correspondence between bulk fields and dual operators?

are not well understood.

Results

The main results we find are:

- The dual theory is a deformation of a d-dimensional CFT.
- The deformation is **irrelevant** w.r.t. relativistic conformal group.
- The deformation is **exactly marginal** w.r.t. **non-relativistic** conformal group.
- The theory is non-local in the v direction.

The small b limit

In the small b limit the geometry is a small perturbation of AdS and standard AdS/CFT applies.

- Massive vector model:

$$S_{CFT} \to S_{CFT} + \int d^d x \, b^i X_i$$

$\to$ X_i has dimension $(d+1)$ and is dual to the bulk vector field.

$\to$ b^i is a null vector with only non-zero component $b^v = b$.

The small b limit

- Topologically massive gravity:

$$S_{CFT} \to S_{CFT} + \int d^2x \, b^{ij} X_{ij}$$

  - $\to$ X_{ij} has dimension $(3, 1)$; it is related to the additional boundary condition associated with the 3rd order equations of TMG. (van Rees, Skenderis, M.T. 2009)
  - $\to$ b^{ij} is a null tensor with only non-zero component $b^{vv} = -b^2$.

Schrödinger invariance

- In both cases the non-relativistic scaling dimension of the deformation is
$$\Delta_s = d$$
and so the deformations are marginal wrt this scaling symmetry!
- Next we need to understand what happens at finite b.

Finite b

Bulk perspective:

- Schrödinger solutions solve the complete non-linear equations.

$\rightarrow$ The theory is Schrödinger invariant for any b.

Boundary QFT perspective:

- We analyzed this question using conformal perturbation theory.

$\rightarrow$ The deforming operator is indeed exactly marginal wrt Schrödinger.

Exact marginality

To explain this computation we need a few facts about theories with Schrödinger invariance:

- Operators are labeled by their non-relativistic scaling dimension, Δ_s and their charge under $\mathcal{M}$, the mass operator.
- In our context the mass operator is the lightcone momentum k_v.
- Operators with different k_v are considered as independent operators.
- In our case, the deforming operator has zero lightcone momentum, $k_v = 0$.

Exact marginality

To prove that the operator is exactly marginal it suffices to show that its 2-point function does not receive any corrections when we turn on b.

$$\langle X_v(k_v{=}0, u_1, x_1^i) X_v(k_v{=}0, u_2, x_2^i)\rangle_{\mathbf{b}} = \langle X_v(k_v{=}0, u_1, x_1^i) X_v(k_v{=}0, u_2, x_2^i)\rangle_{\mathbf{b=0}}$$

This can be studied using conformal perturbation theory.

Conformal perturbation theory

One can show that

$$\langle X_v(k_v) \prod_{i=1}^{n} b^\mu \cdot X_\mu(k_v{=}0) X_v(-k_v) \rangle_{\rm CFT} =$$
$$\langle X_v(k_v) X_v(-k_v) \rangle_{\rm CFT} \, (b^v k_v)^n f(\log k_v, \ldots)$$

where $f(\log k_v, \ldots)$ is a dimensionless function that depends at most polynomially on $\log k_v$.

- Taking the limit $k_v \to 0$, establishes that $X_v(k_v{=}0)$ is exactly marginal.
- The dimensions of operators with $k_v \neq 0$ receive corrections,

$$\Delta_s = \Delta_s(b = 0) + \sum_{n>0} \mathbf{c_n} (b k_v)^n$$

Summary

- We started with a relativistic CFT and deformed it by an irrelevant operator which is however marginal from the perspective of the Schrödinger group.
- We showed that the deformation is exactly marginal and the deformation takes the theory from a relativistic fixed point to a non-relativistic one.

Summary

- The question is then to understand the spectrum of operators in the new fixed point.
- We have seen that in the non-relativistic dimension Δ_s of operators with $k_v \neq 0$ changes as we go from one fixed point to the other.
- We will next analyze this question from the bulk perspective.

Probe scalar

Let us start by analyzing a probe scalar field in the Schrödinger background,

$$S = -\frac{1}{2}\int d^3x\sqrt{-G}\Big(\partial_\mu \Phi \partial^\mu \Phi + m^2\Phi^2\Big).$$

The field equations are

$$\ddot{\Phi} + 2\dot{\Phi} + \Box_\zeta \Phi - (m^2 - b^2\partial_v^2)\Phi = 0$$

The asymptotics of the solution are

$$\Phi = e^{(\Delta_s - 2)r}\Big(\phi_{(0)}(k) + \ldots + e^{-(2\Delta_s - 2)r}\phi_{(2\Delta_s - 2)}(k) + \ldots\Big)$$

Probe scalar

- The dual operator has dimension

$$\Delta_s = 1 + \sqrt{1 + m^2 + b^2 k_v^2}$$

- For small b it takes the form we found earlier using conformal perturbation theory

$$\Delta_s = \Delta_s(b = 0) + \sum c_n (b k_v)^n$$

where $\Delta_s(b = 0) = 1 + \sqrt{1 + m^2}$ is the standard holographic formula for the dimension of a scalar operator.

Correlation functions

- To compute correlation functions we need to compute the on-shell value of the action.
- This suffers from the infinite volume divergences.
- Adapting holographic renormalization we find that we need counterterms

$$S_{\mathrm{ct},\Delta_s \lesssim 3} = -\frac{1}{2}\int d^2k\,\sqrt{-\zeta}\Big((\Delta_s - 2)\Phi^2 + \frac{k_\zeta^2\Phi^2}{2\Delta_s - 4}\Big)$$

- When $b = 0$ these reduce to the counterterms for the scalar field in AdS.

Non-locality

$$S_{\mathrm{ct},\Delta_s \lesssim 3} = -\frac{1}{2}\int d^2k\,\sqrt{-\zeta}\Big((\Delta_s - 2)\Phi^2 + \frac{k_\zeta^2\Phi^2}{2\Delta_s - 4}\Big)$$

- Because Δ_s depends on k_v, the counterterms are not polynomials in k_v.
- The theory is non-local in the v direction.

2-point function

- Having determined the counterterms, the 2-point function can now extracted from an exact solution of the linearized field equations[1]:

$$\langle \mathcal{O}_{\Delta_s}(u, k_v)\mathcal{O}_{\Delta_s}(0, -k_v)\rangle = c_{\Delta_s,k_v}\delta_{\Delta,\Delta_s}u^{-\Delta_s},$$

  where c_{Δ_s,k_v} is a (specific) normalization factor.
- This is precisely of the expected form for a 2-point function of a Schrödinger invariant theory [Henkel (1993)].

[1] Suppressing real-time issues considered in [Leigh-Hoang, Blau et al (2009)]

Gravitational sector

We now turn to the gravitational sector and discuss the solutions to the linearized equations around the background.

- Both models (massive vector and TMG) admit two distinct sets of solutions to the linearized equations.
- The 'T' solutions are associated with the dual stress energy tensor.
- The 'X' solutions are associated with the dual deforming operator.

‘X’ solutions: TMG

- The mode satisfies a hypergeometric equation.
- The dimension of the dual operator is

$$\Delta_s(X_{vv}) = 1 + \sqrt{1 + b^2 k_v^2}$$

  This has the correct limit as $b \to 0$.
- The linearized solution is more singular at the boundary than the Schrödinger background. This is due to the fact that the operators with $k_v \neq 0$ are irrelevant.
- The 2-point function takes the expected form.

‘T’ solutions

The ‘T’ mode perturbations take the form:

$$
\begin{aligned}
h^T_{uu} &= \frac{1}{r^2} h_{(-2)uu} + \tilde{h}_{(0)uu} \log(r^2) + h_{(0)uu} + r^2 h_{(2)uu} \\
h^T_{uv} &= \frac{1}{r^2} h_{(-2)uv} + \tilde{h}_{(0)uv} \log(r^2) + h_{(0)uv} + r^2 h_{(2)uv} \\
h^T_{vv} &= h_{(0)vv} + r^2 h_{(2)vv},
\end{aligned}
$$

- These modes at $b = 0$ reduce to the modes that couple to the energy momentum tensor, T_{ij}.
- The solution is more singular than the Schrödinger background. This is because $\Delta_s(T_{uu}) = 4$ and thus this operator is irrelevant (from the perspective of Schrödinger).

Stress energy tensor

Subtleties in understanding this sector:

- In a **non-relativistic theory** the tensor that contains the conserved energy and momentum is not symmetric and therefore cannot couple to any metric mode.
- This tensor couples instead to the vielbein $\rightarrow$ formulate holography as a Dirichlet problem for the vielbein.
- Part of stress energy tensor is irrelevant, so source must be treated perturbatively.

A long story....!

Null dipole theory

- [Maldacena et al (2008)] argued that the massive vector model in $d = 4$ is dual to a null dipole theory, a non-local deformation of $N = 4$ SYM.
- In the null dipole theory, the ordinary product is replaced by a non-commutative product that depends on a null vector [Ganor et al (2000)]. Expressed in terms of ordinary products the null dipole theory contains terms that:
  - irrelevant from the relativistic CFT point of view
  - marginal from the Schrödinger perspective

  $\rightarrow$ This is in exact agreement with our findings.

Conclusions

The dual to Schrödinger and "null warped" backgrounds is

- a deformation of a d-dimensional CFT.
- The deformation is **irrelevant** w.r.t. relativistic conformal group.
- The deformation is **exactly marginal** w.r.t. **non-relativistic** conformal group.
- The theory is non-local in the v direction.

Analogous story for dynamical exponents $z \neq 2$...

Outlook

- Very little is currently known about the null dipole theories: is the divergence structure the same as we found in gravity?
- How does the dipole theory resum the series in b to produce square roots $\Delta = 1 + \sqrt{1 + b^2 k_v^2}$ in operator dimensions?
- Understand better the stress energy sector and the correct notion of asymptotically Schrödinger.
- How does the anisotropic theory reproduce entropy of warped AdS_3 black holes? (Mysteriously, Cardy formula used by Strominger worked even though dual is not a CFT!)