Kaluza-Klein Consistency, Pauli Reductions

and the Breathing Mode

Gauge Theories and the Structure Of Spacetime

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17’th September 2010

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Based on work with Duff, Nilsson, Warner; Stelle; Cvetič, Lü, Gibbons, Tran, Sadrzadeh;...; and Jim Liu.

- Our most promising candidates for providing a consistent theory of quantum gravity, and for unifying all the fundamental forces in nature, begin life in higher dimensions; superstrings in $D = 10$ and M-Theory in $D = 11$.

- For phenomenological reasons (we only perceive four spacetime dimensions), and for more abstract theoretical reasons (e.g. the AdS/CFT correspondence), it is important to be able to perform dimensional reductions to give lower-dimensional theories from the $D = 10$ or $D = 11$ starting points.

- The key idea for how to do this was conceived by Theodor Kaluza, in 1926. The idea was refined by Oskar Klein in the 1930's and is now known as *Kaluza-Klein Reduction*.

- In its original form, Kaluza's reduction amounted to taking a five-dimensional theory (Einstein gravity), with coordinates $x^M = (x^\mu, y)$, ($M = 0,1,2,3,4$, $\mu = 0,1,2,3$), and then assuming that the five-dimensional fields were independent of the fifth coordinate y. (i.e. $\partial/\partial y$ is a Killing vector.)

- Klein added the physically-motivated notion that the fifth dimension should be wrapped up into a circle of small radius L, in order to account for its non-observability.

- Reduction of $D = 5$ Einstein gravity yields 4-dimensional Einstein-Maxwell gravity with a scalar field. The $U(1)$ gauge transformation of A_μ is associated with general-coordinate transformations $y \longrightarrow y + \xi(x^\mu)$ on the circle.

- The generalisation to reduction on an n-dimensional torus (product of circles) is immediate. This torus is called the *Internal Space*. Its $U(1)^n$ isometry implies $U(1)^n$ gauge fields in the lower dimension.

- Pauli, in 1953, was the first to suggest generalising the Kaluza-Klein idea to a reduction on a *curved* internal space, in order to get *nonabelian* gauge fields. His idea was to reduce six-dimensional Einstein gravity to four dimensions, by using a small 2-dimensional sphere as the internal space. Its $SO(3)$ isometry would imply $SO(3)$ gauge fields (Yang-Mills) in four dimensions. He also realised that this wouldn't work (i.e it is inconsistent).

- Ten years later, in 1963, DeWitt observed that non-abelian gauge fields could be obtained by reduction on a *group manifold* G. This has $G \times G$ isometry, but by only asking for the gauge fields of one copy of G, the reduction procedure is guaranteed to be *consistent*. DeWitt's idea is clever, but not subtle.

- This talk is going to describe how supergravity has found a very subtle way to solve Pauli's original problem. But, it requires delicate "conspiracies," and even small changes can be risky...

Reduction on S^1

Consider a free massless scalar in $D+1$ dimensions, satisfying $\widehat{\Box}\widehat{\phi}(x^\mu, y) = 0$. Fourier expanding on the circle $0 \le y \le 2\pi L$, i.e. $\widehat{\phi}(x,y) = \sum_n \phi_n(x) e^{iny/L}$, implies the D-dimensional fields ϕ_n have masses $m^2 = n^2/L^2$:

$$\Box\phi_n - \frac{n^2}{L^2}\,\phi_n = 0\,.$$

One massless field ϕ_0 plus an infinite tower of massive fields. The usual Kaluza-Klein idea is to truncate to the massless sector.

For gravity, we Fourier expand $\widehat{g}_{MN}(x,y)$. Essentially, this gives towers of spin-2, spin-1 and spin-0 fields $g^{(n)}_{\mu\nu}(x)$, $g^{(n)}_{\mu y}(x)$ and $g^{(n)}_{yy}(x)$, which are massive except for the $n=0$ modes.

Actually, in the massless truncation, a nicer way to parameterise the metric is

$$d\widehat{s}^2 = e^{2\phi}\, g_{\mu\nu}(x) dx^\mu dx^\nu + e^{-2(D-2)\phi}\,(dy - A_\mu dx^\mu)^2\,.$$

The $(D+1)$-dimensional Einstein-Hilbert Lagrangian $\widehat{\mathcal{L}} = \sqrt{-\widehat{g}}\widehat{R}$ reduces to give the D-dimensional Lagrangian

$$\mathcal{L} = \sqrt{-g}\Big(R - (D-1)(D-2)(\partial\phi)^2 - \tfrac{1}{4} e^{-2(D-1)\phi} F^{\mu\nu}F_{\mu\nu}\Big)\,.$$

Kaluza-Klein Consistency

- A crucial point in the S^1 reduction is that it is *consistent* to truncate out the infinite towers of massive fields.
- Imagine substituting the full Fourier mode expansions into the (highly non-linear) $(D+1)$-dimensional Einstein equations. The general form of the equation of motion for a field $\Phi^{(n)}(x)$ in the expansions will be

$$\Box\Phi^{(n)} - \frac{n^2}{L^2}\Phi^{(n)} = \sum_p \Phi^{(n+p)}\Phi^{(-p)} + \sum_{p,q}\Phi^{(n+p+q)}\Phi^{(-p)}\Phi^{(-q)} + \cdots$$

- To be able to set a given field $\Phi^{(n)}$ to zero, there must be no source terms on the RHS that could force it to be non-zero.
- If we make a truncation to set **all** fields $\Phi^{(n)}$ with $n \neq 0$ to zero, and keep **all** fields with $n = 0$, then consistency is guaranteed.
- In other words, massless fields (uncharged under $U(1)$) cannot act as sources for massive fields (charged under $U(1)$).
- A equivalent way to define a *Consistent Truncation* is one for which any solution of the lower-dimensional theory lifts back to a solution of the higher-dimensional theory.

DeWitt Reduction

Here, the higher-dimensional theory is reduced on a compact group manifold G. Has isometry group $G\times G$. We denote this by $G_L\times G_R$, with G_L, G_R acting on the left and the right:

$$g \longrightarrow g_L\, g\, g_R\,.$$

We now make a generalised Fourier expansion of the higher-dimensional fields in terms of complete sets of harmonics on G. In the lower dimension, the massless fields will therefore include the Yang-Mills gauge bosons of $G_L\times G_R$.

In general, a truncation that retains all the $G_L\times G_R$ gauge bosons will be *inconsistent* (but see later). By contrast, in the DeWitt reduction, only the gauge bosons of one copy of G, say G_R, are retained in the truncation.

To be precise, in the DeWitt reduction, the ony fields that are retained are the full set of fields that are **singlets under** G_L.

This is manifestly a *consistent* reduction: non-linear products of G_L singlets cannot act as sources for G_L non-singlets.

Because G_L acts transitively on the group manifold G, there will only be a *finite number of fields* in the truncation. From the higher-dimensional metric, we get the metric, G_R gauge bosons, and a certain set of scalar fields.

Pauli Reduction

Now let's consider a *Pauli Reduction*, where the internal space is taken to be a coset manifold G/H, such as the n-sphere, which is $SO(n+1)/SO(n)$. Reductions of this type are the most important in string theory and M-theory; for example type IIB reduced on S^5, or M-theory reduced on S^7.

In general, Pauli reductions don't work. Only in very special cases, because of remarkable conspiracies between the properties of the higher-dimensional theory, and harmonics on the sphere.

To see the problem in general, consider the example of Einstein gravity. The relevant terms in the metric reduction ansatz:

$$d\hat{s}^2 = g_{\mu\nu}(x) dx^\mu dx^\nu + g_{mn}(y)(dy^m - K_I^m A_\mu^I dx^\mu)(dy^n - K_J^n A_\nu^J dx^\nu)$$

where g_{mn} is the metric on the coset space G/H, K_I^m are the Killing vectors of the isometry group G, and A_μ^I are the Yang-Mills gauge potentials for the group G.

The problem shows up in the lower-dimensional components of the higher-dimensional Einstein equation, which give:

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = \tfrac{1}{2} K_I^m K_{mJ} (F_{\mu\rho}^I F_\nu^{J\rho} - \tfrac{1}{4} F_{\rho\sigma}^I F^{J\rho\sigma} g_{\mu\nu})$$

This would be fine if $K_I^m K_{mJ}$ was equal to δ_{IJ}. But it isn't...

Pauli's Problem

Take Pauli's example of $S^2 = SO(3)/SO(2)$ reduction. We have three Killing vectors of $SO(3)$, on the two-dimensional sphere. So

$$Y_{IJ} \equiv K_I^m K_{mJ}$$

is a 3×3 matrix of rank 2, so it cannot possibly be δ_{IJ}.

Much worse than this, Y_{IJ} is not constant: it depends on the coordinates y^m of S^2. So the "Einstein equation" in the lower dimension is inconsistent: the "energy-momentum tensor" of the gauge fields depends on the coordinates of the compactifying sphere.

What is really happening is that the G gauge bosons are trying to act as sources not only for the usual massless graviton that we are keeping in the Pauli reduction, but also for certain *massive* gravitons that we wanted to truncate away. Thus the truncation is an *inconsistent* one.

The same problem generically arises for *any* choice of coset space G/H.

One of the seemingly attractive advantages of Pauli over DeWitt, namely G gauge bosons from only $\dim(G) - \dim(H)$ extra dimensions, rather than $\dim(G)$, is actually part of its downfall.

Supergravity Rescues Pauli

We illustrated the key "Pauli Problem" with a pure gravity example, but it would arise in just the same way in the reduction of any generic theory. Supergravities, however, are rather special, and sometimes they can conspire to solve the consistency problem.

Consider type IIB supergravity compactified on $S^5 = SO(6)/SO(5)$. Actually, we need only consider the metric and self-dual 5-form:

$$\hat{R}_{MN} = \tfrac{1}{96}\hat{H}_{MPQRS}\hat{H}_N{}^{PQRS}\,, \qquad \hat{H}_5 = \hat{*}\hat{H}_5\,, \qquad d\hat{H}_5 = 0$$

As well as the previous metric reduction ansatz, we now need

$$\hat{H}_5 = (\mathbf{1} + \hat{*})(4g\,\epsilon_5 - \frac{1}{2g}{*F^I} \wedge dK_I)$$

This gives an extra contribution on the RHS of the Einstein equation, so

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} - 2g^2 g_{\mu\nu} = \tfrac{1}{2}Y_{IJ}(F^I_{\mu\rho}F^{J\rho}_\nu - \tfrac{1}{4}F^I_{\rho\sigma}F^{J\rho\sigma}g_{\mu\nu})$$

where now

$$Y_{IJ} = K^m_I K_{mJ} + \frac{1}{2g^2}(\nabla^{[m}K^{n]}_I)(\nabla_{[m}K_{n]J})$$

Y_{IJ} is a 15×15 matrix of rank $5 + 10 = 15$, and it *is* constant, and hence with appropriate scalings $Y_{IJ} = \delta_{IJ}$. Pauli lives!

Consistency of Supergravity Pauli Reductions

- The "Killing vector conspiracy" is one necessary condition for being able to perform a consistent Pauli reduction. Almost all Pauli reductions will fail the test. If a particular example, such as type IIB supergravity reduced on S^5 passes the test, there are still many more hurdles before one can claim to have proven the consistency of the reduction.

- (Note that when examining the Killing vector conspiracy, we omitted, for simplicity, certain scalar fields that *must* be included for full consistency. The crucial point is that the particular consistency test we were applying *would not be affected by including the scalars*.)

- There are very strong indications that we can in fact obtain five-dimensional $\mathcal{N}=8$ gauged $SO(6)$ supergravity by the S^5 reduction. But there is no complete proof, and there is little understanding of *why* it should work.

- Consistency of the S^7 reduction of $D=11$ supergravity is essentially proved, but why it works is equally mysterious.

- A very few other examples of consistent Pauli reductions are known: S^4 reduction of $D=11$ supergravity; S^2 reduction of $D=6$ Salam-Sezgin supergravity; reduction of bosonic string theory on group manifold G, keeping all the $G_L \times G_R$ gauge bosons.

Pauli Reduction with Breathing Mode?

- Since we don't understand why Pauli reductions (occasionally) work, it could be of interest to try modifying, or extending, the rare examples that do work, to see if any further remarkable conspiracies can take place.

- It was recently suggested by Gauntlett, Kim, Varela and Waldram that it might be possible to extend the consistent S^5 or S^7 reductions to maximal gauged $D = 5$ or $D = 4$ supergravity by adding certain *massive* $\mathcal{N} = 8$ supermultiplets, including the one containing the breathing mode of the sphere. (The scalar field parameterising the overall volume factor.)

- This would be remarkable, if true, and would provide a counterexample to standard gravity "lore" that one cannot have a consistent coupling of massive spin-2 fields to gravity.

- The motivation GKVW provided for this proposal can be described as follows. The $SO(6)$ isometry group of S^5 has an $SU(3)$ subgroup that still acts transitively on S^5. Take the infinite towers of $SO(6)$ modes in the full S^5 harmonic expansions, and truncate to all the singlets under $SU(3)$. This gives an $\mathcal{N} = 2$ gauged supergravity in $D = 5$, coupled to a finite number of fields including the breathing mode. (This reduction, however, is *guaranteed* by group theory to be consistent, and does not require any miracles.)

Breathing Mode Destroys Pauli Consistency

It is nevertheless interesting to see what happens if we add in the breathing mode in the previous Pauli reduction. The metric ansatz is now

$$
\begin{aligned}
d\hat{s}^2 &= e^{5\phi} g_{\mu\nu}(x) dx^\mu dx^\nu \\
&\quad + e^{-3\phi} g_{mn}(y)(dy^m - K_I^m A_\mu^I dx^\mu)(dy^n - K_J^n A_\nu^J dx^\nu)
\end{aligned}
$$

For the 5-form, consistency with the Bianchi identity requires

$$
\hat{H}_5 = (\mathbf{1} + \hat{*})(4g e^{20\phi} \epsilon_5 - \frac{1}{2g} e^{4\phi} {*F^I} \wedge dK_I)
$$

The five-dimensional components of the ten-dimensional Einstein equation now imply

$$
R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} - 2g^2 g_{\mu\nu} = \tfrac{1}{2} W_{IJ} (F_{\mu\rho}^I F_\nu^{J\rho} - \tfrac{1}{4} F_{\rho\sigma}^I F^{J\rho\sigma} g_{\mu\nu})
$$

where

$$
W_{IJ} = e^{-8\phi} K_I^m K_{mJ} + \frac{1}{2g^2} e^{4\phi} (\nabla^{[m} K_I^{n]})(\nabla_{[m} K_{n]J})
$$

In the absence of the breathing mode (i.e. $\phi = 0$), the y dependence of the first term cancels that of the second, yielding just δ_{IJ}. But now, when $\phi \neq 0$, the y dependence fails to cancel, and we have an inconsistency.

What Have We Learned?

- The original Kaluza-Klein idea, of reduction on a circle, provides a powerful tool for consistent dimensional reduction, and is guaranteed to work. Solutions of the lower-dimensional theory can always be lifted back as solutions in the higher dimension.

- The extension to reduction on an n-torus is immediate. DeWitt extended the idea further, by taking the internal space to be any compact group manifold G. Although the isometry group is $G_L \times G_R$, in the DeWitt reduction only fields that are singlets under G_L are retained. In particular, this means keeping just the non-abelian gauge bosons of G_R. *The consistency of the DeWitt reduction is trivially guaranteed by group theory*.

- The highly non-trivial Pauli reductions in supergravity, such as the S^5 reduction of type IIB, are much more interesting because they are totally mysterious. We have very little understanding of why they work.

- The consistent Pauli reductions are very delicate, and any modifications or extensions seem likely to destroy the consistency. We examined one particular example, and showed that a proposal to add the breathing mode in the reduction renders the S^5 Pauli reduction inconsistent. (The same happens in the S^7 reduction of $D = 11$ supergravity.)