Extended supersymmetry and geometry

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Outline

- Sigma models
- Geometry of Supersymmetric sigma models
- Relation to generalized Kähler geometry
- Gerbes
- Gerbes and the generalized Kähler potential
- Biholomorphic Gerbes
- Spinors on $T \oplus T^*$
- Supergravity solutions
- Construction from the sigma model data

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Based on

"Generalized Kahler manifolds and off-shell supersymmetry".
Commun.Math.Phys.269:833-849,2007
"Generalized Kahler geometry and gerbes,"
JHEP **0910**, 062 (2009) , [arXiv:0811.3615 [hep-th]] .
"Generalized Calabi-Yau metric and Generalized Monge-Ampère equation".
JHEP **1008**, 060, (2010) . [arXiv:1005.5658 [hep-th]]

with

Chris Hull,
Martin Roček,
Rikard von Unge,
Maxim Zabzine.

Sigma models

$$\phi^i : \Sigma \to \mathcal{T}$$

$$S = \int_\Sigma d\phi^i G_{ij}(\phi) \star d\phi^j$$

$$\nabla^2 \phi^i := \partial^2 \phi^i + \partial \phi^j \Gamma_{jk}{}^i \partial \phi^k = 0$$

$$S = \int_{\Sigma_B} d\xi \left\{ \eta^{\mu\nu} \partial_\mu X^i G_{ij}(X) \partial_\nu X^j + \ldots \right\}$$

SUSY sigma models

Ex. ($d = 2, N = (2,2)$ chiral fields)

$$\{D_\alpha, \bar{D}_\beta\} = 2i\partial_{\alpha\beta}$$

$$\phi(z) \to \phi(z, \theta) :$$

$$X(z) = \phi| \ , \quad \Psi_\alpha(z) = D_\alpha \phi| \ , \quad F(z) = D^2\phi|$$

$$S \to \int dz d\bar{z} D^2 \bar{D}^2 \ K(\phi, \bar{\phi})$$

$$= \int dz d\bar{z} (\partial X \ G_{X\bar{X}}(X, \bar{X}) \bar{\partial} \bar{X} + ...)$$

where

$$G_{X\bar{X}}(X,\bar{X}) = \partial_X \partial_{\bar{X}} K(X,\bar{X})$$

$\Longleftrightarrow$ $\mathcal{T}$ carries Kähler Geometry

$$\boxed{\text{Susy } \sigma \text{ models} \quad \Longleftrightarrow \quad \text{Geometry of } \mathcal{T}}$$

d=	6	4	2	Geometry
N=	1	2	4	Hyperkähler
N=		1	2	Kähler
N=			1	Riemannian

Message

- Supersymmetric sigma models provide a powerful tool to probe complex geometry.
- The more supersymmetries, the more specialized the geometry

- Additional supersymmetries, when examined at the $(2,2)$ level, lead to interesting new structures on the target space.

The (1,1)-D-algebra:

$$\mathbb{D}_{\pm}^{2} = i\partial_{\substack{+\\=}}$$

$$\mathcal{S} \;=\; \int d^2x\mathbb{D}_+\mathbb{D}_-\left(\mathbb{D}_+\varphi^i(G_{ij} + B_{ij})\mathbb{D}_-\varphi^j\right) .$$

The (1,1) analysis by Gates, Hull and Roček gives:

Susy	(0,0) (1,1)	(2,2)	(2,2)	(4,4)	(4,4)
Bgd	G, B	G	G, B	G	G, B
Geom	Riem.	Kähler	biherm.	hyperk.	bihyperc.

Ansatz for the extra supersymmetries:

$$\delta\varphi^i = \epsilon^+ J^i_{(+)j}\mathbb{D}_+\varphi^j + \epsilon^- J^i_{(-)j}\mathbb{D}_-\varphi^j$$

Invariance of the action and closure of the algebra requires the geometry to be bi-hermitean: Data $(M, G, J_\pm, H)$

$$J^2_{(\pm)} = -\mathbf{1}$$

$$J^t_{(\pm)} G J_{(\pm)} = G$$

$$N(J_{(\pm)}) = 0$$

$$\nabla^{(\pm)} J_{(\pm)} = 0 \ , \quad \Gamma^{(\pm)} = \Gamma^0 \pm \tfrac{1}{2} G^{-1} H \ , \quad H := dB$$

$$H \simeq H J_{(\pm)} J_{(\pm)}$$

An alternative description

Data: Bihermitean $(M, G, J_{\pm})$ with integrability conditions

$$
\begin{aligned}
d^c_{(+)}\omega_{(+)} + d^c_{(-)}\omega_- &= 0 \\
dd^c_{(\pm)}\omega_{(\pm)} &= 0 \; .
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_{(\pm)} &:= GJ_{(\pm)} \\
d^c &:= i(\bar{\partial} - \partial) \; ,
\end{aligned}
\tag{1}
$$

and

$$
H = H^{(2,1)} + H^{(1,2)} = d^c_{(+)}\omega_{(+)} = -d^c_{(-)}\omega_{(-)} \; .
$$

(2,2) superfields

The (2,2)-D-algebra:

$$\{D_\pm, \bar{D}_\pm\} = 2i\partial_{\pm\!\!\!\pm}$$

Chiral fields ϕ:

$$\bar{D}_\pm \phi = 0 \Rightarrow D_\pm \bar{\phi} = 0$$

Twisted chiral fields χ:

$$\bar{D}_+ \chi = D_- \chi = 0 \Rightarrow D_+ \bar{\chi} = \bar{D}_- \bar{\chi} = 0$$

Left/Right semi-chiral fields $\mathbb{X}_{L/R}$:

$$\begin{aligned} \bar{D}_+ \mathbb{X}_L = 0 \Rightarrow D_+ \bar{\mathbb{X}}_L = 0 \\ \bar{D}_- \mathbb{X}_R = 0 \Rightarrow D_- \bar{\mathbb{X}}_R = 0 \end{aligned} \qquad (2)$$

These are all the fields needed.

Relation to GKG

$$\begin{aligned} \mathcal{S} &= \int d^2x D^2 \bar{D}^2 K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_{L/R}, \bar{\mathbb{X}}_{L/R}) \\ &\to \int d^2x \left(\partial_{+\!\!+} \varphi^i (G_{ij} + B_{ij}) \partial_= \varphi^j + \ldots.. \right). \end{aligned}$$

$$(J_{(\pm)},\ G, H = dB)\ ,$$

$$J^2_{(\pm)} = -1, \qquad N(J_{(\pm)}) = 0, \quad [J_{(+)}, J_{(-)}] \neq 0\ ,$$

$$J^t_{(\pm)} G J_{(\pm)} = G, \quad H = d^c_{(+)} \omega_{(+)} = -d^c_{(-)} \omega_{(-)}$$

A complete description of GKG.*

* Locally and away from irregular points.

A Gerbe outline

$$\boxed{\text{Line Bundle}}$$

$$\downarrow$$

$$\boxed{\text{Line Bundle with connection}}$$

$$\downarrow$$

$$\boxed{\text{Holomorphic Line Bundle (with connection)}}$$

$$\boxed{\text{Gerbe}}$$

$$\downarrow$$

$$\boxed{\text{Gerbe with connection}}$$

$$\downarrow$$

$$\boxed{\text{Holomorphic Gerbe (with connection)}}$$

$$\downarrow$$

$$\boxed{\text{Biholomorphic Gerbe (with connection)}}$$

Gerbes

Maps defined on each threefold intersection

$$g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U\gamma \to S^1$$

satisfying

$$g_{\alpha\beta\gamma} = g_{\beta\gamma\alpha} = g_{\gamma\alpha\beta} = g^{-1}_{\beta\alpha\gamma} = g^{-1}_{\alpha\gamma\beta} = g^{-1}_{\gamma\beta\alpha}$$

as well as a cocycle condition on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

$$g_{\alpha\beta\gamma} g_{\beta\alpha\delta} g_{\gamma\beta\delta} g_{\delta\alpha\gamma} = 1$$

Gerbe
with a connection

$\frac{H}{2\pi} \in H^3(M, \mathbb{Z})$

$$H = dB_\alpha \ ,$$
$$B_\alpha - B_\beta = dA_{\alpha\beta} \ ,$$
$$(\delta A)_{\alpha\beta\gamma} := A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = d\Lambda_{\alpha\beta\gamma} \ ,$$
$$(\delta\Lambda)_{\alpha\beta\gamma\delta} := \Lambda_{\beta\gamma\delta} + \Lambda_{\delta\gamma\alpha} + \Lambda_{\alpha\beta\delta} + \Lambda_{\beta\alpha\gamma} = c_{\alpha\beta\gamma\delta} \ ,$$

where

$$B_\alpha \in \Omega^2(U_\alpha) \ ,$$
$$A_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta) \ ,$$
$$\Lambda_{\alpha\beta\gamma} \in C^\infty(U_\alpha \cap U_\beta \cap U_\gamma) \ ,$$
$$c_{\alpha\beta\gamma\delta} \in 2\pi\,\mathbb{Z} \ .$$

Let $g_{\alpha\beta\gamma} \ : \ U_\alpha \cap U_\beta \cap U_\gamma \ \to \ S^1$ be given by

$$g_{\alpha\beta\gamma} = e^{i\Lambda_{\alpha\beta\gamma}} \ ,$$

This defines a gerbe where $\Lambda_{\alpha\beta\gamma}$ are angles,

$$\Lambda_{\alpha\beta\gamma} \in 2\pi\mathbb{R}/\mathbb{Z} \ .$$

Holomorphic Gerbes

Holomorphic functions

$$G_{\alpha\beta\gamma} \ : \ U_\alpha \cap U_\beta \cap U_\gamma \ \to \ \mathbb{C}^*$$

Hermitean structure:

$$G_{\alpha\beta\gamma}\bar{G}_{\alpha\beta\gamma} = h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}$$

Additional structure

Locally the $(2,1)$-part of H can be written as

$$H^{(2,1)} = i\partial\bar{\partial}\lambda_{\alpha}^{(1,0)} \Rightarrow H = dd^{c}(Re\ \lambda_{\alpha}^{(1,0)})\ .$$

$$B_{\alpha}^{(1,1)} = i\bar{\partial}\lambda_{\alpha}^{(1,0)} - i\partial\bar{\lambda}_{\alpha}^{(0,1)}\ .$$

On $U_{\alpha} \cap U_{\beta}$:

$$\lambda_{\alpha}^{(1,0)} - \lambda_{\beta}^{(1,0)} = \partial\xi_{\alpha\beta} + \phi_{\alpha\beta}^{(1,0)}\ ,$$

where $\phi^{(1,0)}$ is a holomorphic $(1,0)$-form.

Generalized Kähler gauge transformation:

$$K_\alpha - K_\beta$$
$$= F^+_{\alpha\beta}(\phi,\chi,X_L) + \bar{F}^+_{\alpha\beta}(\bar{\phi},\bar{\chi},\bar{X}_L) + F^-_{\alpha\beta}(\phi,\bar{\chi},X_R) + \bar{F}^-_{\alpha\beta}(\bar{\phi},\chi,\bar{X}_R)\ .$$

On $U_\alpha \cap U_\beta \cap U_\gamma$:

$$Re\left(F^+_{\alpha\beta} + F^+_{\beta\gamma} + F^+_{\gamma\alpha} + F^-_{\alpha\beta} + F^-_{\beta\gamma} + F^-_{\gamma\alpha}\right) = 0\ .$$

$\Rightarrow$

$$\begin{aligned} F^+_{\alpha\beta}(\phi,\chi,X_L) + F^+_{\beta\gamma}(\phi,\chi,X_L) + F^+_{\gamma\alpha}(\phi,\chi,X_L) \\ = i\left(c_{\alpha\beta\gamma}(\phi) - b_{\alpha\beta\gamma}(\chi)\right)\ , \\ F^-_{\alpha\beta}(\phi,\bar{\chi},X_R) + F^-_{\beta\gamma}(\phi,\bar{\chi},X_R) + F^-_{\gamma\alpha}(\phi,\bar{\chi},X_R) \\ = -i\left(c_{\alpha\beta\gamma}(\phi) + \bar{b}_{\alpha\beta\gamma}(\bar{\chi})\right)\ . \end{aligned}$$

c and b are (twisted) biholomorphic functions $f : (d - id^c_{\pm})f = 0$;

$$c_{\alpha\beta\gamma} = -c_{\beta\alpha\gamma} = -c_{\alpha\gamma\beta} = -c_{\gamma\beta\alpha} \ ,$$
$$b_{\alpha\beta\gamma} = -b_{\beta\alpha\gamma} = -b_{\alpha\gamma\beta} = -b_{\gamma\beta\alpha} \ .$$

On $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$:

$$(\delta c)_{\alpha\beta\gamma\delta} - (\delta b)_{\alpha\beta\gamma\delta} = 0 \ ,$$
$$(\delta c)_{\alpha\beta\gamma\delta} + (\delta \bar{b})_{\alpha\beta\gamma\delta} = 0$$

$\Rightarrow$

$$c_{\beta\gamma\delta} + c_{\delta\gamma\alpha} + c_{\alpha\beta\delta} + c_{\beta\alpha\gamma} = \frac{i}{4} d_{\alpha\beta\gamma\delta} \ ,$$
$$b_{\beta\gamma\delta} + b_{\delta\gamma\alpha} + b_{\alpha\beta\delta} + b_{\beta\alpha\gamma} = \frac{i}{4} d_{\alpha\beta\gamma\delta} \ .$$

$^*\Rightarrow \quad d_{\alpha\beta\gamma\delta} \in 2\pi\mathbb{Z}$.

$$
\begin{aligned}
\lambda_+ &= -i\left(\frac{\partial K}{\partial X_R^{\alpha'}}dX_R^{\alpha'} + \frac{\partial K}{\partial \phi^a}d\phi^a - \frac{\partial K}{\partial \chi^{a'}}d\chi^{a'}\right) + c.c. \\
\lambda_- &= i\left(\frac{\partial K}{\partial X_L^{\alpha}}dX_L^{\alpha} + \frac{\partial K}{\partial \phi^a}d\phi^a + \frac{\partial K}{\partial \chi^{a'}}d\chi^{a'}\right) + c.c.
\end{aligned}
$$

$$
\begin{aligned}
\xi^+_{\alpha\beta} &= i(\bar{F}^- - F^-)_{\alpha\beta} \\
\phi^{(1,0)}_{+\alpha\beta} &= i(F^+_{a'}d\chi^{a'} - F^+_a d\phi^a)_{\alpha\beta} \\
\xi^-_{\alpha\beta} &= i(F^+ - \bar{F}^+)_{\alpha\beta} \\
\phi^{(1,0)}_{-\alpha\beta} &= i(F^-_a d\phi^a - F^-_{\bar{a}'} d\bar{\chi}^{\bar{a}'})_{\alpha\beta}
\end{aligned}
$$

$\Rightarrow$

$$
\Lambda_{\alpha\beta\gamma} = i\left(\bar{c}_{\alpha\beta\gamma}(\bar{\phi}) - c_{\alpha\beta\gamma}(\phi) + \bar{b}_{\alpha\beta\gamma}(\bar{\chi}) - b_{\alpha\beta\gamma}(\chi)\right).
$$

Biholomorphic:

$$G_{\alpha\beta\gamma}(\phi) = e^{4c_{\alpha\beta\gamma}(\phi)}~, \qquad F_{\alpha\beta\gamma}(\chi) = e^{4b_{\alpha\beta\gamma}(\chi)}~,$$

$$G_{\alpha\beta\gamma}~, F_{\alpha\beta\gamma}~:~U_\alpha \cap U_\beta \cap U_\gamma~, \rightarrow~\mathbb{C}^*~,$$

Antisymmetric under permutations of the open sets and satisfy the cocycle condition on the four-fold intersection.

In addition,

$$G_{\alpha\beta\gamma}F^{-1}_{\alpha\beta\gamma} = h^+_{\alpha\beta}h^+_{\beta\gamma}h^+_{\gamma\alpha} \ , \qquad G_{\alpha\beta\gamma}\bar{F}_{\alpha\beta\gamma} = h^-_{\alpha\beta}h^-_{\beta\gamma}h^-_{\gamma\alpha} \ ,$$

where $h^\pm_{\alpha\beta} = exp(\mp 4iF^\pm_{\alpha\beta})$ special $J_\pm$-holomorphic function.

Compare to Hermiticity for a holomorphic line-bundle:

$$G_{\alpha\beta}\bar{G}_{\alpha\beta} = e^{K_\alpha}e^{-K_\beta} \ ,$$

where we may interpret K as the Kähler potential.

• We have not been able to retrieve the generalized Kähler potential from the hermiticity conditions of a biholomorphic gerbe in all cases.

- Want to relate the Type II supergravity solutions for metric, dilaton and NS-flux to the world-sheet description in terms of $N = (2,2)$ sigma models.

Spinors ρ on $T \oplus T^*$:
A section $X + \xi$ of $T \oplus T^*$ acts on a form ρ:

$$(X + \xi) \cdot \rho = \imath_X \rho + \xi \wedge \rho$$

Invariant bilinear form (Mukai pairing):

$$(\rho_1, \rho_2) = \sum_j (-1)^j [\rho_1^{2j} \wedge \rho_2^{d-2j} + \rho_1^{2j+1} \wedge \rho_2^{d-2j-1}] \, ,$$

A spinor ρ is *pure* if it annihilates a maximal isotropic subspace of $T \oplus T^*$.

A *generalized Kähler structure*:
Two commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that the quadratic form $\langle \mathcal{J}_1 \mathcal{J}_2 (X + \xi), (X + \xi) \rangle$ is positive definite.

A *generalized Calabi-Yau metric structure*:
A pair of closed pure spinors ρ_1 and ρ_2 such that the corresponding generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ give rise to a generalized Kähler structure and $(\rho_1, \bar{\rho}_1) = \alpha(\rho_2, \bar{\rho}_2)$ for some non-zero constant α.

A supergravity solution

The *Gualtieri map* gives us $g_{\mu\nu}, H_{\mu\nu\rho}$ and thus the relation to the sigma model. The dilaton Φ comes from normalization of the spinors

$$(\rho_1, \bar{\rho}_1) = \alpha(\rho_2, \bar{\rho}_2) = e^{-2\Phi}\mathrm{vol}_g = e^{-2\Phi}\sqrt{g}\, dx^1 \wedge \ldots \wedge dx^D \ .$$

The data $(g_{\mu\nu}, H_{\mu\nu\rho}, \Phi)$ is a Type II supersymmetric supergravity solution. It automatically solves the equation

$$R^+_{\mu\nu} + 2\nabla^-_\mu \partial_\nu \Phi = 0 \ ,$$

Construction from the sigma model

Ansatz:

$$\rho_{1,2} = N_{1,2} \wedge e^{R_{1,2}+iS_{1,2}} \ ,$$

where

$$\begin{aligned}
N_1 &= e^{f(\phi)} d\phi^1 \wedge \ldots \wedge d\phi^{d_c} \ , \\
N_2 &= e^{g(\chi)} d\chi^1 \wedge \ldots \wedge d\chi_{d_t} \ , \\
R_1 &= -d(K_L dX_L) \ , \\
R_2 &= -d(K_R dX_R) \ , \\
S_1 &= d(K_T J d\chi + K_L J dX_L - K_R J dX_R) \ , \\
S_2 &= -d(K_C J d\phi + K_L J dX_L + K_R J dX_R) \ ,
\end{aligned}$$

These are pure spinors with the correct properties.

The Generalized Monge-Ampère equation

$$(\rho_1, \rho_1) = \alpha(\rho_2, \rho_2) \implies$$

$$(-1)^{d_s d_c} e^{f(\phi)} e^{\bar{f}(\bar{\phi})} \det \begin{pmatrix} -K_{l\bar{l}} & -K_{lr} & -K_{l\bar{t}} \\ -K_{\bar{r}\bar{l}} & -K_{\bar{r}r} & -K_{\bar{r}\bar{t}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{t}} \end{pmatrix}$$

$$= \alpha \; e^{g(\chi)} e^{\bar{g}(\bar{\chi})} \det \begin{pmatrix} K_{l\bar{r}} & K_{l\bar{l}} & K_{l\bar{c}} \\ K_{r\bar{r}} & K_{r\bar{l}} & K_{r\bar{c}} \\ K_{c\bar{r}} & K_{c\bar{l}} & K_{c\bar{c}} \end{pmatrix}$$

$$e^{2\Phi} = (-1)^{d_s d_c} \frac{e^{-f(\phi)} e^{-\bar{f}(\bar{\phi})}}{det\ K_{LR}} \det \begin{pmatrix} -K_{l\bar{l}} & -K_{lr} & -K_{l\bar{t}} \\ -K_{\bar{r}\bar{l}} & -K_{\bar{r}r} & -K_{\bar{r}\bar{t}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{t}} \end{pmatrix}$$