Higher Derivative Gravity and Entanglement Entropy

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Introduction

- The AdS/CFT correspondence provides a tool for studying large N_c gauge theories at strong coupling. Has been applied to several problems of interest from nuclear physics to condensed matter (chiral symmetry breaking, viscosity to entropy ratio, marginal fermi liquid description, superconductors etc.)

- Interesting to study higher derivative gravity theories in the context of the AdS/CFT correspondence. They provide a holographic example where $\mathrm{c} \neq \mathrm{a}$.

Introduction

Gravitational theories with higher derivative terms in general

- Have ghosts when expanded around flat space.

- Their equations of motion contain more than two derivatives of the metric. Hard to solve exactly.

  Additional degrees of freedom.

  In holography, this implies the existence of extra fields in the boundary CFT.

  [Skenderis, Taylor and van Rees].

Introduction

There exists a special class of gravitational theories with higher derivative terms, Lovelock gravity.

$$S = \int d^{d+1}x\sqrt{-g}\sum_{p=0}^{[\frac{d}{2}]}(-)^p\frac{(p-2d)!}{(p-2)!}\lambda_p\mathcal{L}_p$$

with $[\frac{d}{2}]$ the integral part of $\frac{d}{2}$, λ_p are the Lovelock parameters and the p-th order Lovelock term $\mathcal{L}_p$ is

$$\mathcal{L}_p = \frac{1}{2^p}\delta^{\mu_1\nu_1\cdots\mu_p\nu_p}_{\rho_1\sigma_1\cdots\rho_p\sigma_p}R^{\rho_1\sigma_1}{}_{\mu_1\nu_1}\cdots R^{\rho_p\sigma_p}{}_{\mu_p\nu_p}$$

$\mathcal{L}_p$ is the Euler density term in $2p$–dimensions.

Introduction

We choose $\lambda_0 = 1$ and $\lambda_1 = -1$ such that

$$\mathcal{L}_0 = \frac{d(d-1)}{L^2} \qquad\qquad \mathcal{L}_1 = R\,.$$

Examples:

- **2nd order Lovelock term $\Leftrightarrow$ Gauss-Bonnet**

$$\mathcal{L}_2 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$$

- **3rd order Lovelock term**

$$\begin{aligned}\mathcal{L}_3 \;=\; & 2R^{\rho\sigma\kappa\lambda}R_{\kappa\lambda\mu\nu}R^{\mu\nu}{}_{\rho\sigma} + 8R^{\rho\sigma}{}_{\kappa\mu}R^{\kappa\lambda}{}_{\sigma\nu}R^{\mu\nu}_{\rho\lambda} + \\ + \; & 24R^{\rho\sigma\kappa\lambda}R_{\kappa\lambda\sigma\mu}R^{\mu}_{\rho} + 3RR^2_{\rho\sigma\kappa\lambda} + 24R^{\rho\kappa\sigma\lambda}R_{\sigma\rho}R_{\lambda\kappa} + \\ + \; & 16R^{\rho\sigma}R_{\sigma\kappa}R^{\kappa}_{\rho} - 12RR^2_{\rho\sigma} + R^3\end{aligned}$$

Introduction

Special Properties of the Lovelock action:

- **Equations of motion contain only up to second order derivatives of the metric $\Rightarrow$ No additional boundary data.**
- **No ghosts when expanded around Minkowski flat background.**
- **Palatini and Metric formulations equivalent [Exirifard, Sheikh–Jabbari].**

Introduction

Study Lovelock theories of gravity in the context of the AdS/CFT correspondence. Are there any signs that we can trust holography in this case? If so, what new features does the boundary CFT acquire, given the additional parameters of the theory λ_p? Can they teach us something new?

Objective of this talk:

Focus on holographic entanglement entropy. New features and tests.

Work in progress with Jan de Boer and Andrei Parnachev.

Outline

- Entanglement Entropy: A review
- Holographic Description of Entanglement Entropy
- EE in four dimensional CFTs : Solodukhin's Result
- Fursaev's proposal and Generalizations
- Holographic calculations of EE in Gauss-Bonnet gravity
- Summary, Conclusions and Open Questions

Entanglement Entropy: Review

- **Consider a quantum mechanical system at zero temperature in a pure state $|\Psi\rangle$. The density matrix is $\rho_0 = |\Psi\rangle\langle\Psi|$ and the von Neumman entropy vanishes**

$$S = -\mathrm{tr}\rho_0 \ln \rho_0 = 0\,.$$

- **"Divide" the system into two subsystems A, B with Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$. The reduced density matrix $\rho_A = \mathrm{tr}_B \rho_0$ is accessible only to A. The entanglement entropy for the subsystem A is the von Neumman entropy of the reduced density matrix ρ_A**

$$S_A = -\mathrm{tr}_A \rho_A \ln \rho_A$$

Entanglement Entropy: Review

The entanglement entropy, EE, measures how "quantum" a system is.

Example:

Consider two systems A, B with Hilbert spaces consisting of two states $\{|1\rangle, |2\rangle\}$. The total Hilbert space is the product of the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$.

Product State:

$$|1_A 1_B\rangle \Rightarrow S_A = 0$$

Pure (non product) State:

$$\frac{1}{\sqrt{2}}(|1_A 2_B\rangle - |2_A 1_B\rangle) \Rightarrow S_A = \ln 2$$

Entanglement Entropy: Review

EE satisfies a number of different properties:

- **For the subsystem V and its complement V^c entanglement entropy is equal.**

$$S(V) = S(V^c)$$

- **For any two subsystems A, B entanglement entropy satisfies the strong subadditivity property**

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B)$$

Entanglement Entropy: Review

EE in a continuous system is UV divergent. The "Area Law" of EE refers to the form of the leading divergence

$$S(V) \sim \frac{Area(\partial V)}{\epsilon^{d-2}} + \cdots$$

Note: **The "Area Law" is violated for systems with a Fermi surface [Wolf, Gioev, Klich, ...].**

For a conformal field theory, CFT, in d-dimensions

$$S(V) = \frac{g_{d-2}[\partial V]}{\epsilon^{d-2}} + \cdots + \frac{g_1[\partial V]}{\epsilon} + g_0[\partial V] \ln \epsilon + s(V)\,.$$

If V has a single characteristic length scale, R, $g_i[\partial V]$ is a homogeneous function of degree i of R.

Entanglement Entropy: Review

Functions $g_i[\partial V]$ with $i \neq 0$ are non-physical, cutoff dependent.

- **The coefficient of the logarithmically divergent term in the EE, $g_0[\partial V]$, is physical and universal.**

In 2-dimensional CFTs the leading divergent term is logarithmic. Its coefficient is proportional to the central charge c of the CFT.

e.g: The EE of a line segment of length l

$$S(l) = \frac{c}{3} \ln \frac{l}{\epsilon}$$

[Casini, Huerta]: An alternative proof of the c-theorem in combining this result with the strong subadditivity property of EE.

Entanglement Entropy: Review

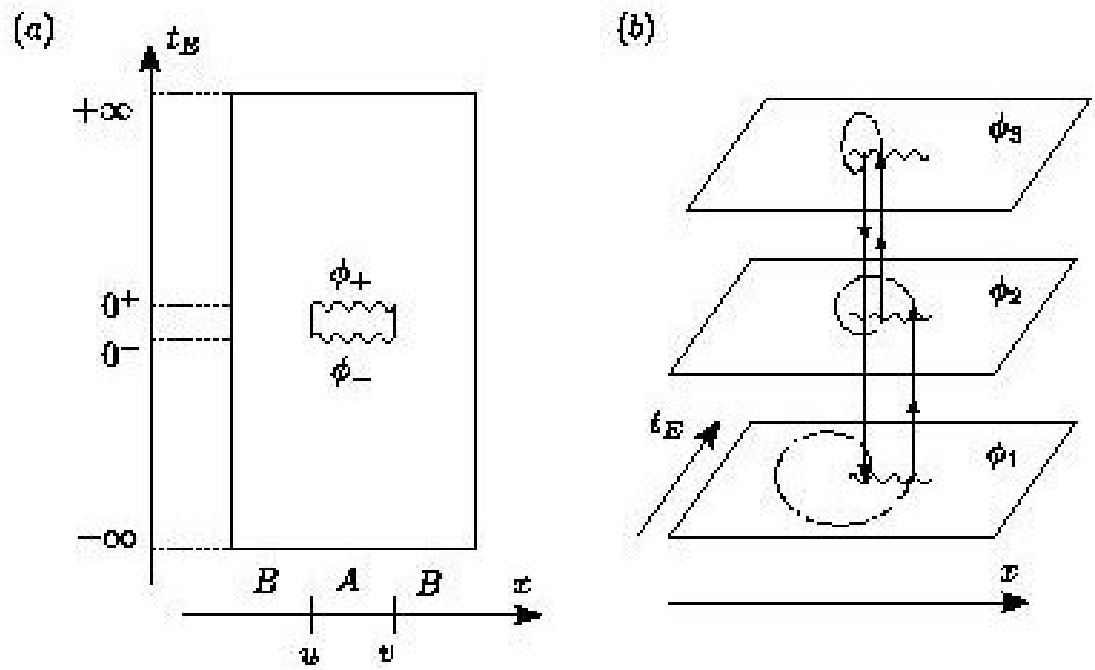
- **How to compute EE in quantum field theory?**

The replica trick:

$$S(V) = \lim_{n\to 1} \frac{\mathrm{tr}_V \rho_V^n - 1}{1-n} = -\frac{\partial}{\partial n} \ln \mathrm{tr}_V \rho_V^n|_{n=1}$$

In the path integral formalism $\mathrm{tr}_V \rho_V^n = \frac{Z_n}{Z_1^n}$ **and one computes the partition function** Z_n **by gluing together** n **copies of** $\mathbb{R}_d$ **along the boundary** (∂V)**.**

Entanglement Entropy: Review



(a) Path integral representation of the reduced density matrix,

(b) The n-sheeted surface, with $n = 3$ for simplicity.

Holographic Description of EE

[Ryu-Takayanagi]

The EE in a CFT on $\mathbf{R}_d$ of a subspace V with arbitrary $(d-2)$-dimensional boundary $(\partial V) \in \mathbf{R}_{d-1}$ is given by

$$S(V) = \frac{1}{4G_N^{(d+1)}} \int_\Sigma \sqrt{\sigma}$$

Here Σ is the static d-dimensional minimal surface within AdS$_{d+2}$ which asymptotes to (∂V).

The proposal has been generalized to non-conformal cases and the near horizon limit of Dp–branes. A covariant formulation has been proposed as well.

[Ryu, Takayanagi, Klebanov, Kutasov, Murugan, Hubeny, Rangamani]

Holographic Description of EE

Ryu-Takayanagi formula passed several tests:

- It is trivially equal for V and its complement V^c (when evaluated at zero temperature).
- At zero temperature, in the limit of very large V the holographic EE vanishes. At finite temperature it asymptotes to the thermal entropy.
- Satisfies the strong subadditivity property.

  [Headrick, Takayanagi]
- Agreement with field theoretic results in 2-dimensional CFTs [Calabrese, Cardy].

Solodukhin's result for EE in $4d$-CFTs.

The coefficient of the logarithmic term in the EE of a subspace V with boundary ∂V of extrinsic curvature $k^i_{\mu\nu}$

$$g_0[\partial V] = \frac{\mathbf{c}}{720\pi} g_{0c}[\partial V] - \frac{\mathbf{a}}{720\pi} g_{0a}[\partial V]$$

$\mathbf{c}$, $\mathbf{a}$ are the CFT central charges defined through the Weyl anomaly on a curved manifold

$$\langle T^\mu_\mu \rangle = \frac{1}{90} \times \frac{1}{64\pi^2} \left(\mathbf{c} \mathrm{I}_2 - \mathbf{a} \mathcal{L}_{(2)} \right)$$

I_2 is the square of the Weyl tensor and $\mathcal{L}_{(2)}$ is the Euler density in four dimensions, *i.e.*, the Gauss–Bonnet term.

Solodukhin's result for EE in $4d$-CFTs.

g_{0c}, g_{0a} **depend on the details of the boundary** ∂V

$$g_{0c}[\partial V] = \int_{\partial V} R_{\mu\nu\sigma\tau}(n_i^\mu n_i^\sigma)(n_j^\nu n_j^\tau) - R_{\mu\nu} n_i^\mu n_i^\nu + \frac{1}{3}R + \\ + \int_{\partial V} \left[\frac{1}{2}k^i k^i - (k^i_{\mu\nu})^2\right]$$

$$g_{0a}[\partial V] = \int_{\partial V} R_{(\partial V)}$$

- n_i **with** $i = 1, 2$ **are vectors normal to the surface** (∂V)
- $k^i_{\mu\nu}$ **is the extrinsic curvature associated to** n^i**.**

Solodukhin's result for EE in $4d$-CFTs.

Corollary for the EE of any four dimensional CFT:

- **For V a ball $\mathcal{B}$ of radius of R**

$$g_0(\mathcal{B}) = \frac{\mathbf{a}}{90}$$

- **For V a cylinder $\mathcal{C}$ of radius R and "infinite" length l**

$$g_0(\mathcal{C}) = \frac{\mathbf{c}}{720}\frac{l}{R}$$

Solodukhin's result for EE in $4d$-CFTs.

Solodukhin's result for the coefficient of the logarithmically divergent term in the entanglement entropy of a ball was confirmed for the case of a free massless scalar field both numerically and analytically.

[Lohmayer, Neuberger, Schwimmer, Theisen / Casini, Huerta]

Note: **This result provides a** *new, distinct* **characterization of the central charges $(\mathbf{c}, \mathbf{a})$ of the CFT.**

Connection to Zamolodchikov's theorem? Generalization to arbitrary dimensions?

Fursaev's proposal and Generalizations

In all CFTs dual to Einstein-Hilbert gravity (with a cosmological constant): $\mathbf{a = c}$.

- **Is there a way to distinguish between the two central charges in holography?**

Gauss-Bonnet gravity, **is a higher derivative gravity with this property.**

$$S_{GB} = \frac{1}{16\pi G_N^{(5)}} \int d^5x \sqrt{-g} \left(R + \frac{12}{L^2} + \frac{\lambda_{GB} L^2}{2} \mathcal{L}_{(2)} \right)$$

Fursaev's proposal and Generalizations

Gauss-Bonnet gravity admits two AdS solutions. One solution is unstable against small perturbations.

Consider the stable solution with radius:

$$L_{AdS}^2 = \frac{1+\sqrt{1-4\lambda_{GB}}}{2} L^2$$

Computation of the Weyl anomaly for Gauss-Bonnet gravity determines the CFT central charges in terms of the Gauss-Bonnet parameter λ_{GB} [Nojiri, Odintsov].

$$\mathbf{c} = 45\pi \frac{L_{AdS}^3}{G_N^{(5)}} \sqrt{1-4\lambda_{GB}}$$

$$\mathbf{a} = 45\pi \frac{L_{AdS}^3}{G_N^{(5)}} \left[-2+3\sqrt{1-4\lambda}\right]$$

Fursaev's proposal and Generalizations

A proposal for holographic EE in Gauss-Bonnet gravity [Fursaev].

$$S(V) = \frac{1}{4G_N^{(5)}} \int_\Sigma \sqrt{\sigma} \left(1 + \lambda_{GB} L^2 R_\Sigma\right)$$

Σ is the minimal surface ending on (∂V) which satisfies the e.o.m. derived from this action. R_Σ is the induced scalar curvature on Σ.

- **Equal to Wald's entropy formula.**
- **Satisfies all of the properties of EE, including strong subadditivity [Headrick, Takayanagi].**

Holographic EE for a ball of radius R

Finding the exact minimal surface is a difficult problem. Solving for the leading divergent terms in the EE is easy.

Consider the case of a ball. Write the AdS metric as

$$ds^2_{AdS} = L^2_{AdS}\left[\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(-dt^2 + dr^2 + r^2 d\Omega_2^2\right)\right]$$

Symmetries indicate that Σ is determined by a single function $r(\rho)$. The e.o.m. in the vicinity of the boundary $\rho = 0$ are solved by

$$r(\rho) = R - \frac{\rho}{2R} + \cdots$$

Substitute into the "action" to arrive at

$$S(B) = \frac{\mathbf{a}}{90}\frac{R^2}{\epsilon^2} + \frac{\mathbf{a}}{90}\ln\epsilon + \cdots$$

Holographic EE for a cylinder and a belt

In similar manner, the leading divergent terms in the EE of an infinite cylinder are

$$S(C) = \frac{\mathbf{a}}{90}\frac{2\pi R l}{4\pi\epsilon^2} + \frac{\mathbf{c}}{720}\frac{l}{R}\ln\epsilon + \cdots$$

For an infinite belt of width y the induced scalar curvature R_Σ vanishes. The result only differs from that of Ruy-Takayanagi by the substitution of $L \to L_{AdS}$.

$$S(S) = \frac{3\mathbf{c} - \mathbf{a}}{270\pi}\left[\frac{l^2}{\epsilon^2} - 4\pi^{\frac{3}{2}}\left(\frac{\Gamma[\frac{2}{3}]}{\Gamma[\frac{1}{6}]}\right)^3\frac{l^2}{y^2}\right]$$

Interesting to consider for confining backgrounds; qualitative features similar to those observed in [Klebanov, Kutasov, Murugan].

Holographic EE and Lovelock gravity

- **Holographic results from Fursaev's proposal in perfect agreement with Solodukhin's.**

A natural generalization of Fursaev's proposal to any Lovelock theory of gravity

$$S(V) = \frac{1}{4G_N^{(d+1)}} \sum_{p=0}^{[\frac{d}{2}]} (-)^{p+1}(p+1)\frac{(d-2p-2)!}{(d-2)!}\lambda_{p+1}\int_\Sigma \sqrt{\sigma}\mathcal{L}_{(p)}$$

Summary, Conclusions and Open Questions

- Fursaev's formula for the holographic calculation of EE in Gauss-Bonnet gravity agrees with Solodukhin's result.

- There is a natural generalization of this proposal for any Lovelock theory of gravity.

- Phase transition for the EE of a belt in a confining background in Gauss-Bonnet gravity.

Open Questions:

- Generalization of Solodukhin's result to higher dimensional CFTs.

Summary, Conclusions and Open Questions

- **A proposal for EE in any theory of higher derivative gravity.**
- **Helpful perhaps towards finding the analog of Zamolodchikov's theorem in higher dimensions?**