Scattering amplitudes, correlations functions and Wilson loops in gauge theories

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Review + new results obtained in collaboration with

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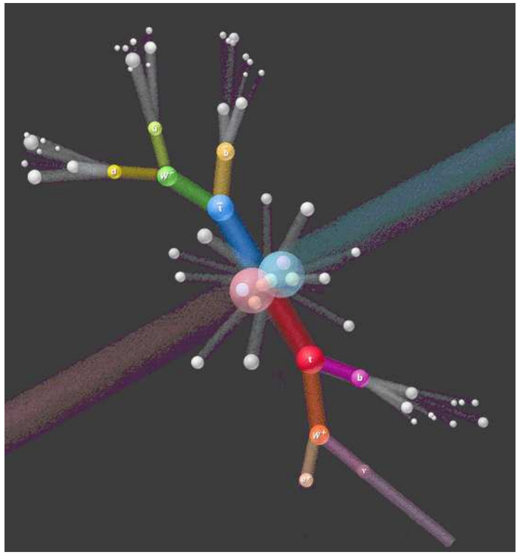
Outline

- Why scattering amplitudes are interesting?
- Hidden symmetries of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory
- Wilson loops/scattering amplitude duality
- Correlation functions/Wilson loops duality
- What happens to these dualities in Yang-Mills theory with less/no supersymmetry
- Recent developments and open questions

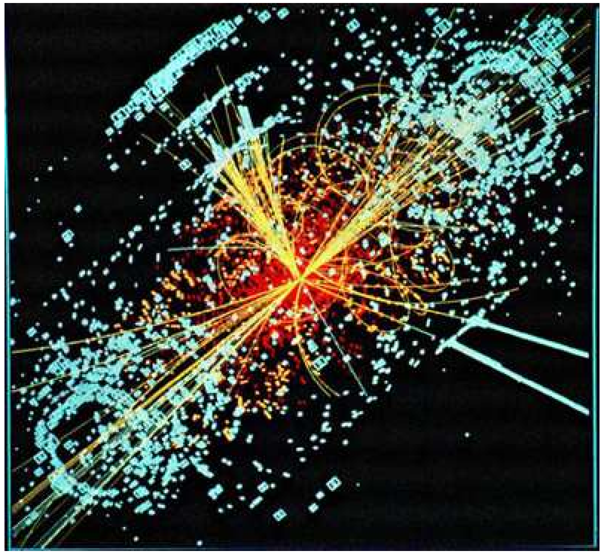
Why scattering amplitudes? Hep-ph motivation:

Search for the Higgs boson at Large Hadron Collider:

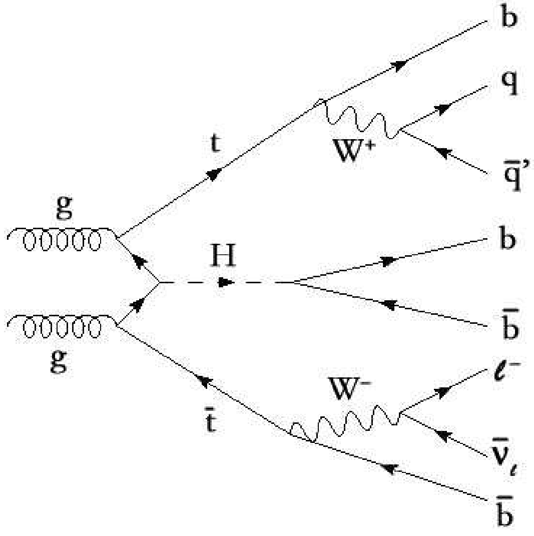
$$\text{gluon} + \text{gluon} \to \text{top quark} + \text{antitop quark} + \text{Higgs}$$



Theory



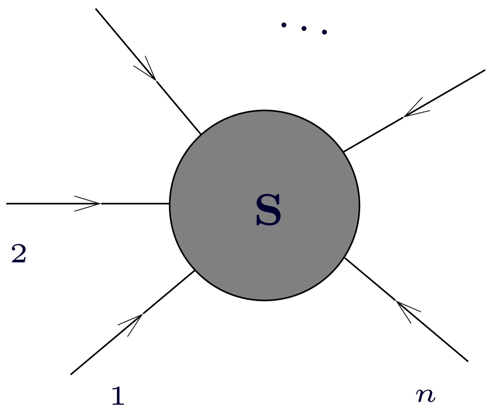
Experiment



Feynman diagram

- ✔ Lots of produced particles in the final state leading to large background
- ✔ Identification of Higgs boson requires detailed understanding of scattering amplitudes
- ✔ Theory should provide solid basis for a successful physics program at the LHC

Why scattering amplitudes? Hep-th motivation:

$\mathcal{A}_n =$ 

- ✔ On-shell matrix elements of $S-$matrix:
  - ✗ Nontrivial functions of Mandelstam variables $S_{ij} = (p_i + p_j)^2$ and YM coupling
  - ✗ Probe (hidden) symmetries of gauge theory
- ✔ In planar $\mathcal{N} = 4$ SYM theory they seem to have a remarkable structure:
  - ✗ Simpler than Standard Model amplitudes but they share many of the same properties
  - ✗ All-order conjectures and a proposal for strong coupling via AdS/CFT
  - ✗ Hints for new symmetry – ***dual superconformal invariance***

General properties of amplitudes in gauge theories

Tree amplitudes:

- ✔ Are well-defined in $D = 4$ dimensions (free from UV and IR divergences)
- ✔ Respect classical (Lagrangian) symmetries of gauge theory
- ✔ *Gluon tree amplitudes* are the same in all gauge theories (QCD, ..., $\mathcal{N} = 4$ SYM)

All-loop amplitudes:

- ✔ Loop corrections are not universal (gauge theory dependent)
- ✔ Free from UV divergences (when expressed in terms of renormalized coupling)
- ✔ Suffer from IR divergences $\to$ are not well-defined in $D = 4$ dimensions
- ✔ Some of the classical symmetries (dilatations, conformal boosts,...) are broken

Three questions in this talk:

- ✔ Do tree amplitudes in $\mathcal{N} = 4$ SYM have hidden symmetries?
- ✔ How powerful are these symmetries to completely determine the scattering amplitudes?
- ✔ What happens to these symmetries in QCD?

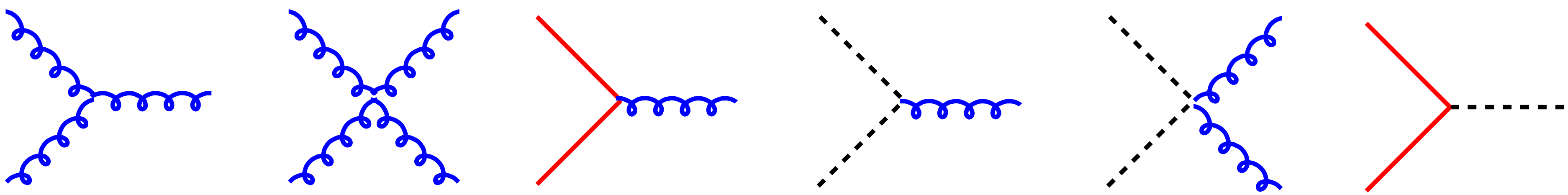
Maximally supersymmetric Yang-Mills theory

- ✔ Uniquely specified by local internal symmetry group - e.g. number of colors N_c for $SU(N_c)$
- ✔ Exactly scale-invariant field theory for any YM coupling g^2
- ✔ Weak/strong coupling duality (AdS/CFT correspondence)

Particle content:

massless spin-1 gluon	(= the same as in QCD)
4 massless spin-1/2 gluinos	(= cousin of the quarks)
6 massless spin-0 scalars	

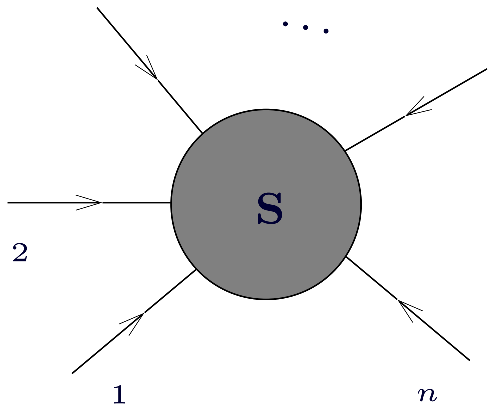
Interaction between particles:



All proportional to same dimensionless coupling g and related to each other by supersymmetry

Gluon amplitudes in $\mathcal{N}=4$ super Yang-Mills theory

- On-shell matrix elements of $S-$matrix:

$\mathcal{A}_n \;=$ 

- Quantum numbers of scattered gluons:

Color:	$a_i = 1,\dots,N_c^2-1$
Light-like momenta:	$(p_i^\mu)^2 = 0$
Polarization state (helicity):	$h_i = \pm 1$

- Color-ordered planar gluon amplitudes:

$$\mathcal{A}_n = \mathrm{tr}\left[T^{a_1}T^{a_2}\dots T^{a_n}\right] A_n^{h_1,h_2,\dots,h_n}(p_1,p_2,\dots,p_n) + \text{[Bose symmetry]}$$

  - Supersymmetry all-loop relations:

$$A_n^{++\dots+} = A_n^{-+\dots+} = 0$$

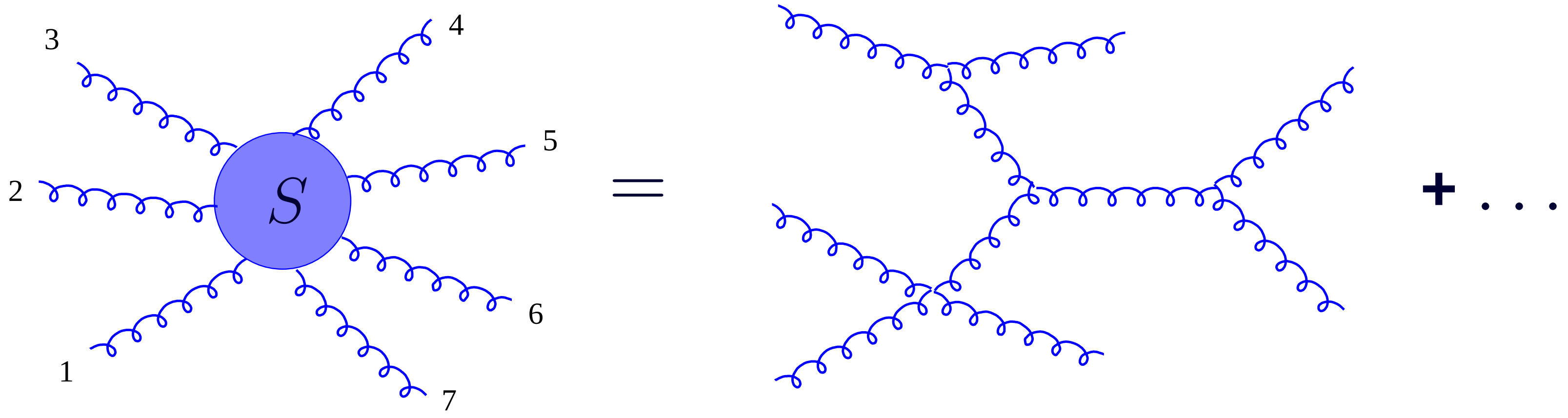
  - Classification of amplitudes according to the total helicity $h=\sum_1^n h_i$

$$\mathrm{MHV} = \{A_n^{--+\dots+}, A_n^{-+-\dots+},\dots\}\,, \qquad \mathrm{NMHV} \quad = \{A_n^{---+\dots+}, A_n^{-+--\dots+},\dots\}$$

MHV = Maximal Helicity Violating ampitudes, NMHV = next-to-MHV, ...

Hints for hidden symmetry

Gluon amplitudes at tree level:



Number of external gluons	4	5	6	7	8	9	10
Number of ‘tree’ diagrams	4	25	220	2485	34300	559405	10525900

- ✔ Number of diagrams grows factorially for large number of external gluons/number of loops
- ✔ ... but the final expression looks remarkably simple (details to come)

$$A_n^{\text{tree}}(1^-2^-3^+\dots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}\delta^{(4)}\Big(\sum_i p_i\Big)$$

Where does this simplicity come from?

Tree MHV amplitude

- ✔ Spinor helicity formalism: [Xu,Zhang,Chang'87]
  - ✗ Parameterization of four-momenta $p^\mu = (p_0, p_1, p_2, p_3)$

$$\hat{p} = p_0 + \sum_{i=1}^{3} p_i \sigma_i = \begin{pmatrix} p_0 + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & p_0 - p_3 \end{pmatrix}$$

  - ✗ On-shell gluon momenta

$$p_\mu^2 = 0 \qquad \Longrightarrow \qquad \det \|\hat{p}\| = 0 \qquad \Longrightarrow \qquad (\hat{p})^{\alpha\dot{\alpha}} = \lambda^\alpha(p)\tilde{\lambda}^{\dot{\alpha}}(p)$$

  - ✗ Commuting spinors: $(\hat{p})^{\dot{\alpha}\alpha}\lambda_\alpha = \tilde{\lambda}_{\dot{\alpha}}(\hat{p})^{\dot{\alpha}\alpha} = 0$:

$$\lambda^\alpha(p) \quad [\text{helicity } -\tfrac{1}{2}]\,, \qquad\qquad \tilde{\lambda}^{\dot{\alpha}}(p) \quad [\text{helicity } +\tfrac{1}{2}]$$

- ✔ Scattering amplitudes are functions of $\lambda_i = \lambda(p_i)$, $\tilde{\lambda}_i = \tilde{\lambda}(p_i)$ $(i = 1, \dots, n)$ [Parke,Taylor]

$$A_n^{\mathrm{MHV}}(1^-2^-3^+\dots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}\delta^{(4)}\Big(\sum_1^n \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}\Big)\,, \qquad \text{(with } \langle ij\rangle \equiv \lambda_i^\alpha \lambda_{j,\alpha}\text{)}$$

What are symmetries of this amplitude?

Conformal symmetry

$\mathcal{N}=4$ SYM has (super)conformal $SU(2,2|4)$ symmetry to all loops:

- ✔ Conformal symmetry acts locally in $x-$space (e.g. inversions $x_\mu \to x_\mu/x^2$)

- ✔ Conformal symmetry is naturally realized on the correlation functions

- ✔ Consequences of conformal symmetry for the correlation functions

$$\langle O_i(x)O_j(0)\rangle = \frac{\delta_{ij}}{(x^2)^{\Delta_i}}$$

$$\langle O_1(x_1)O_2(x_2)O_3(0)\rangle = \frac{C_{123}}{(x_{12}^2)^{\Delta_1+\Delta_2-\Delta_3}(x_2^2)^{\Delta_2+\Delta_3-\Delta_1}(x_1^2)^{\Delta_3+\Delta_1-\Delta_2}}$$

- ✔ Scaling dimensions Δ_i can be determined from integrability of dilatation operator in planar $\mathcal{N}=4$ SYM
  talk by Kazakov

- ✔ Recent progress in computing C_{123} at strong coupling
  talks by Costa,Janik,Tseytlin

Conformal symmetry of the amplitude

- ✔ Conformal symmetry acts locally in $x-$space but non-locally in $p-$space
- ✔ Realization of conformal symmetry for the amplitudes [Witten]

$$k_{\alpha\dot{\alpha}} = \sum_i \frac{\partial^2}{\partial\lambda_i^\alpha \partial\tilde{\lambda}_i^{\dot{\alpha}}} \qquad \Longrightarrow \qquad k_{\alpha\dot{\alpha}}\mathcal{A}_n^{\mathrm{MHV}} = 0$$

In fact, (super)conformal symmetry is almost exact (due to holomorphic anomaly)

$$\bar{s}\,\mathcal{A}_n^{\mathrm{MHV}} \sim \delta^{(2)}(\lambda_i, \lambda_{i+1})\,\mathcal{A}_{n-1}^{\mathrm{MHV}}$$

$\bar{s}\,\mathcal{A}_n^{\mathrm{MHV}}$ is localized at collinear configurations $p_i \| p_{i+1}$ [Bargheer,Beisert,Galleas,Loebbert,McLoughlin]

- ✔ Can be extended to the full $SU(2,2|4)$ superconformal invariance

$$g \cdot \mathcal{A}_n^{\mathrm{MHV}} = 0\,, \qquad g = \{p,\, m,\, k,\, q,\, \bar{q},\, s,\, \bar{s},\, \ldots\} \in SU(2,2|4)$$

Much less trivial to verify for NMHV, N^{2}MHV,... amplitudes [GK,Sokatchev]

- ✔ *Conformal symmetry alone is not powerful enough to fix the tree amplitudes*

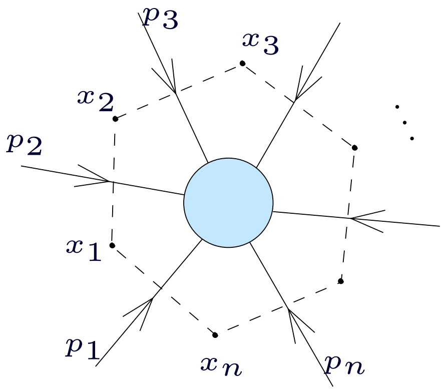
Dual (super)conformal symmetry I

The $\mathcal{N} = 4$ amplitudes have a much bigger, dual conformal symmetry [Drummond, Henn, GK, Sokatchev]

- ✔ Examine absolute value of the amplitude:

$$\left|\hat{A}_n^{\rm MHV}\right|^2 = \frac{(S_{12})^4}{S_{12}S_{23}\dots S_{n1}}\,, \qquad (\text{with} \quad S_{ij} = (p_i + p_j)^2)$$

- ✔ Introduce dual variables (not a Fourrier transform!)



✗ $p_i = x_i - x_{i+1}\,, \quad x_{n+1} \equiv x_1$

✗ $p_i^2 = 0 \Longrightarrow (x_i - x_{i+1})^2 = 0$

✗ $S_{i,i+1} = (x_i - x_{i+2})^2$ whereas
$S_{i,i+2} = (x_i - x_{i+1} - x_{i+2} + x_{i+3})^2$

- ✔ The MHV amplitude in the dual space

$$\left|\hat{A}_n^{\rm MHV}\right|^2 = \frac{[(x_1 - x_3)^2]^3}{(x_2 - x_4)^2(x_3 - x_5)^2\dots(x_n - x_1)^2}$$

Looks like $n-$point correlation function in $x-$space, *BUT x's are the momenta!*

Dual (super)conformal symmetry II

- Conformal inversions in dual $x-$space

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2} \qquad \Longrightarrow \qquad S_{i,i+1} \to \left(x_i^2 x_{i+2}^2\right)^{-1} S_{i,i+1}$$

Acts locally on the momenta $\Longrightarrow$ *is not related* to conformal symmetry of $\mathcal{N}=4$ SYM

- The tree-level MHV (super)amplitude is covariant under dual conformal inversions

$$I\left[\mathcal{A}_n^{\mathrm{MHV}}\right] = \left(x_1^2 x_2^2 \dots x_n^2\right) \times \mathcal{A}_n^{\mathrm{MHV}}$$

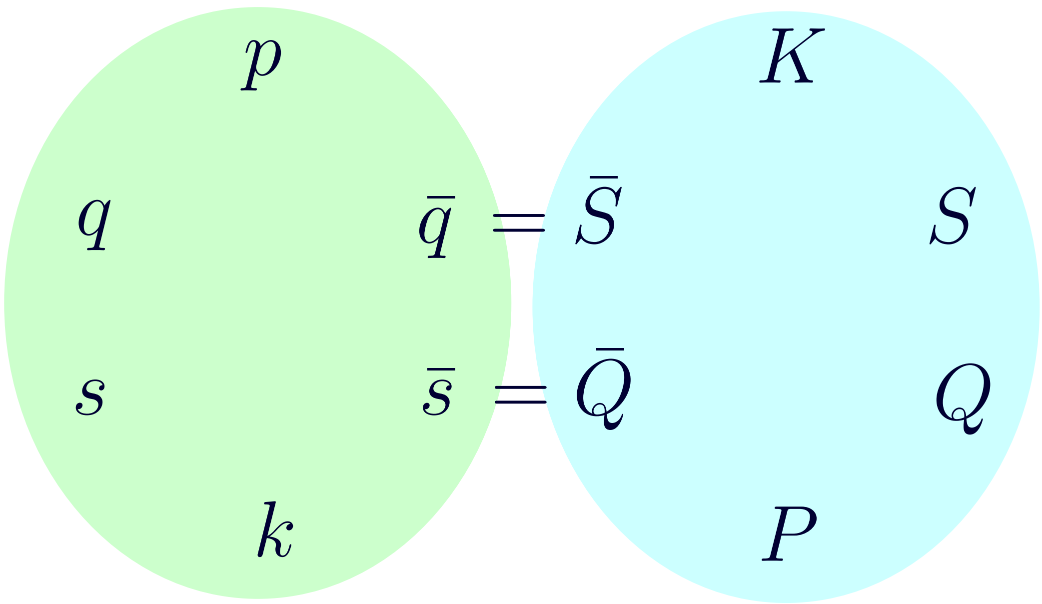
- Dual conformal symmetry can be extended to dual superconformal $\widetilde{SU}(2,2|4)$ symmetry

$$G \cdot \mathcal{A}_n^{\mathrm{MHV}} = 0\,, \qquad G = \{P,\, M,\, K,\, Q,\, \bar{Q},\, S,\, \bar{S},\, \dots\} \in \widetilde{SU}(2,2|4)$$

- ***Dual superconformal symmetry is a property of all tree-level (super)amplitudes (MHV, NMHV, N^{2}MHV,...) in $\mathcal{N}=4$ SYM theory*** [Drummond,Henn,GK,Sokatchev],[Brandhuber,Heslop,Travaglini]

- ✔ The relationship between conventional and dual superconformal $su(2,2|4)$ symmetries:

[Drummond,Henn,GK,Sokatchev]



- ✔ The same symmetries appear at strong coupling from invariance of $\mathrm{AdS}_5 \times \mathrm{S}^5$ sigma model under bosonic [Kallosh,Tseytlin] + fermionic T-duality [Berkovits,Maldacena],[Beisert,Ricci,Tseytlin,Wolf]
- ✔ (Infinite-dimensional) closure of two symmetries has Yangian structure [Drummond,Henn,Plefka]
- ✔ All tree $\mathcal{N} = 4$ amplitudes are uniquely fixed by symmetries + analyticity condition [GK,Sokatchev]

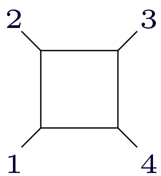
What happens to these symmetries at loop level?

Planar gluon amplitudes at weak coupling

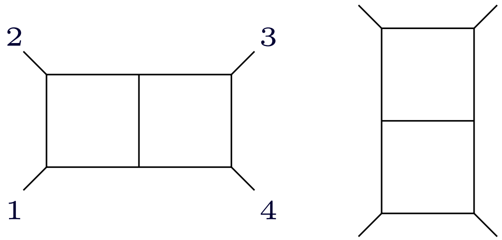
‘Mirracle’ at weak coupling: number of Feynman diagrams increases with loop level but their sum can be expressed in terms of a few ‘special’ scalar box-like integrals

Example: four-gluon amplitude in $\mathcal{N} = 4$ SYM:

- ✔ One loop: [Green,Schwarz,Brink’82]

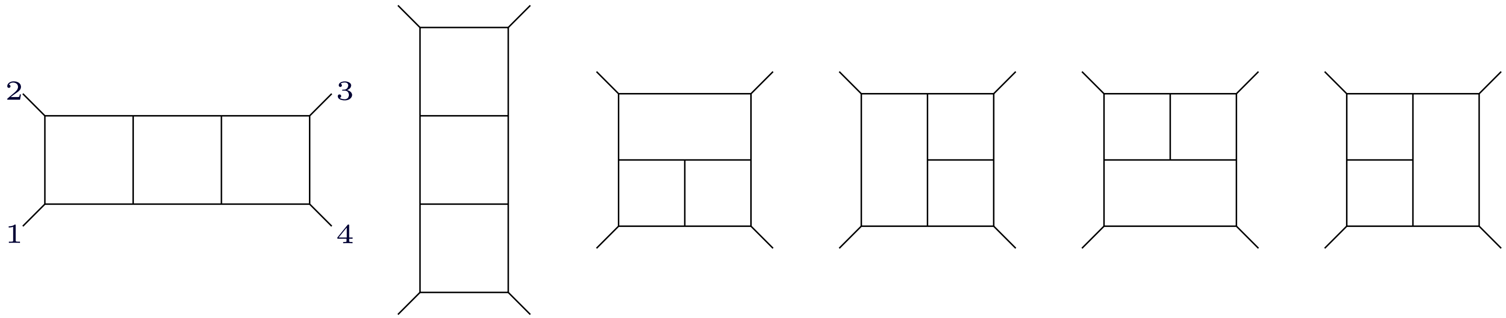


- ✔ Two loops: [Bern,Rozowsky,Yan’97]



  all-loop iteration structure conjectured [Anastasiou,Bern,Dixon,Kosower’03]

- ✔ Three loops: [Bern,Dixon,Smirnov’05]



Little hope to get an exact all-loop analytical solution...

Planar gluon amplitudes at weak coupling II

- ✔ *Loop corrections to all MHV amplitudes are described by a single scalar function* [Parke,Taylor]

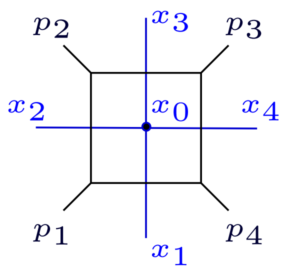
$$A_n^{\rm MHV}(p_i) = A_n^{\rm (tree)}(p_i) M_n^{\rm MHV}(\{s_{ij}\}; a)$$

- ✔ Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling

$$A_4/A_4^{\rm (tree)} = 1 + a\, I^{(1)}(s,t) + O(a^2)\,, \qquad a = \frac{g^2 N_c}{8\pi^2}$$

[Green,Schwarz,Brink]

  - ✗ Dual variables $p_i = x_i - x_{i+1}$ with $p_i^2 = x_{i,i+1}^2 = 0$

$I^{(1)}(s,t) =$ 

$$\sim \int \frac{d^D x_0\, x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$$

  - ✗ Is invariant under conformal transformations, e.g. $x_i \to x_i/x_i^2$, in the ***dual*** space for $D=4$!
  - ✗ The symmetry *is not related* to conformal symmetry of $\mathcal{N}=4$ SYM
  - ✗ All scalar integrals contributing to A_4 up to four loops possess the dual conformal invariance!

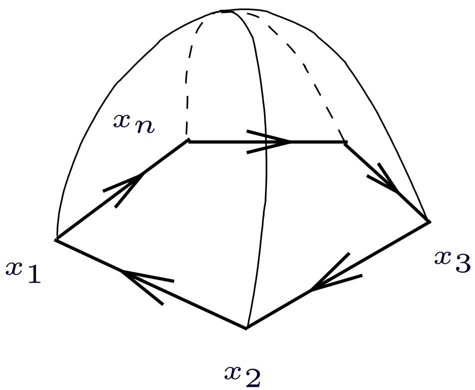
- ✔ The dual conformal symmetry ***is broken by IR divergences***

Guon amplitudes from AdS/CFT

- ✔ At strong coupling, scattering amplitudes are described by a classical string world-sheet in AdS$_5$

[Alday,Maldacena]

$$\mathcal{A}_n \sim \exp\left(-\frac{\sqrt{\lambda}}{2\pi} S_{\min}\right), \qquad ds^2 = \frac{dz_{3+1}^2 + dr^2}{r^2}$$



- ✗ Gluon momenta $p_1^\mu, \ldots, p_n^\mu$ define sequence of light-like segments on the boundary $r = 0$
- ✗ The closed contour has n cusps with the *dual coordinates* x_i^μ

$$x_i^\mu - x_{i+1}^\mu = p_i^\mu$$

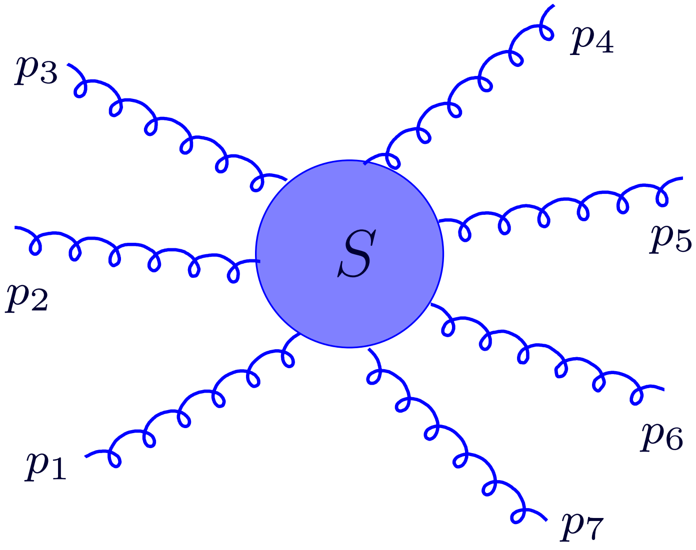
Dual conformal symmetry is present at strong coupling!

- ✔ Agrees with the Bern-Dixon-Smirnov ansatz for $n = 4$ gluon amplitude

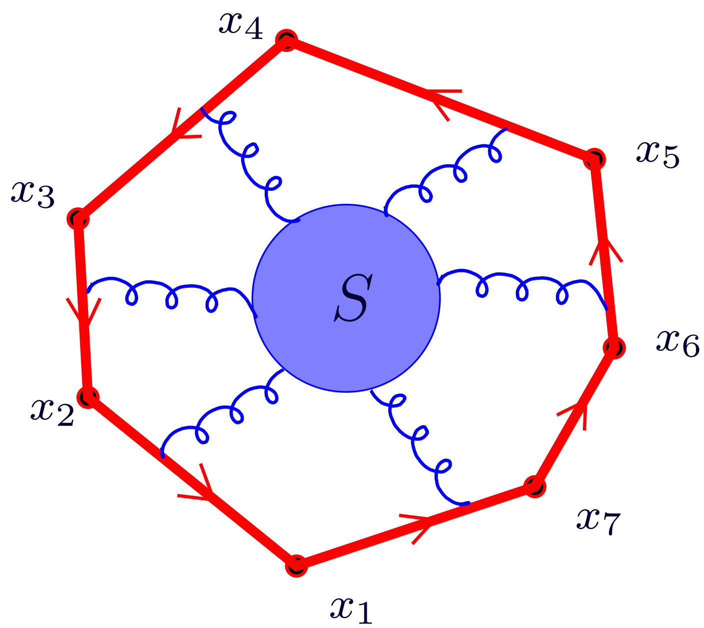
$$\ln \mathcal{A}_4(s,t) = \text{Div}(1/\epsilon) + \frac{1}{4}\Gamma_{\text{cusp}}(a) \ln^2\left(\frac{s}{t}\right) \qquad \text{(BDS ansatz)}$$

- ✔ Calculation of $S_{\min}$ is "mathematically similar" to that of the expectation value of a *cusped* Wilson loop made out of n light-like segments $[x_i,\ x_{i+1}]$ at strong coupling
- ✔ The relation to Wilson loop should hold for arbitrary coupling [Drummond,GK,Sokatchev]

Scattering amplitudes/Wilson loops duality



$\Leftrightarrow$



- *MHV amplitudes are dual to light-like Wilson loops*

$$\ln \mathcal{A}_n^{(\mathrm{MHV})} \sim \ln W(C_n) + O(1/N_c^2)\,, \qquad C_n = \text{light-like } n\text{−(poly)gon}$$

C_n polygon-like contour made of ***gluon momenta*** $p_i = x_i - x_{i+1}$

- At weak coupling, the duality was verified against BDS ansatz to two loops for $n \geq 4$

[Drummond,Henn,GK,Sokatchev], [Anastasiou,Brandhuber,Heslop,Khoze,Spence,Travaglini]

- Dual conformal symmetry of amplitude = Conformal symmetry of Wilson loop

- IR divergences of amplitude = UV divergences of Wilson loop (due to cusps)

Dual conformal anomaly

Dual conformal anomaly $\Leftrightarrow$ Conformal anomaly of Wilson loops

- ✔ Light-like Wilson loops in $\mathcal{N}=4$ SYM

  - ✗ Were the Wilson loop well-defined (=finite) in $D=4$ dimensions it would be conformal invariant

$$W(C_n)=W(C'_n)$$

  - ✗ ... but $W(C_n)$ has cusp UV singularities $\mapsto$ dim.reg. breaks conformal invariance

$$W(C_n)=W(C'_n)\times \text{[cusp anomaly]}$$

- ✔ *All-loop* anomalous conformal Ward identities for the *finite part* of the Wilson loop

$$\ln W(C_n)=F_n^{(WL)}+\text{[UV divergencies]}+O(\epsilon)$$

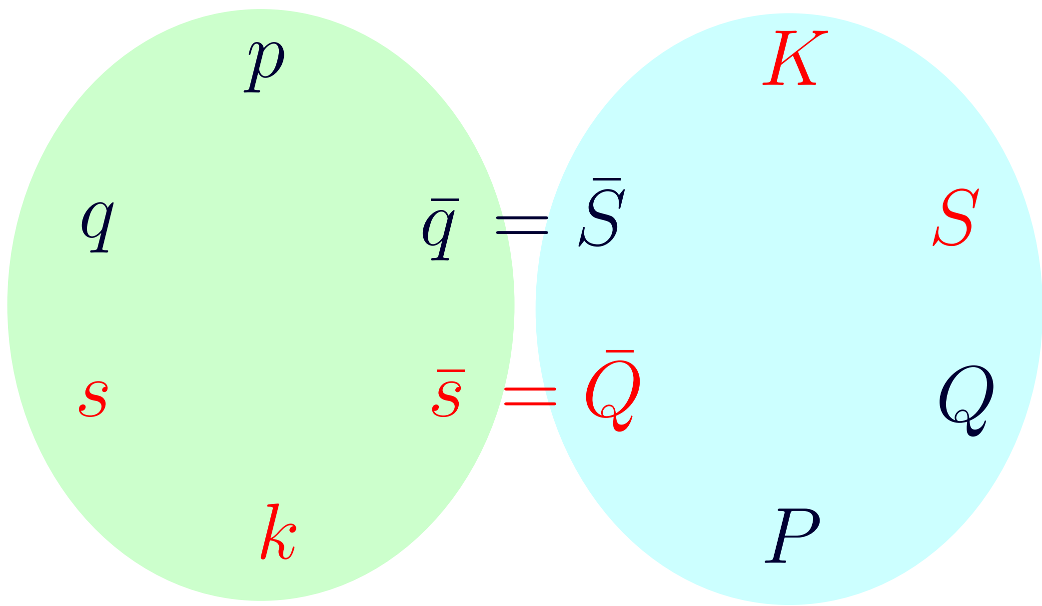
Under special conformal transformations (boosts), to all orders, [Drummond,Henn,GK,Sokatchev]

$$K^\mu\, F_n \equiv \sum_{i=1}^{n}\left[2x_i^\mu(x_i\cdot\partial_{x_i})-x_i^2\partial_{x_i}^\mu\right] F_n = \frac{1}{2}\Gamma_{\rm cusp}(a)\sum_{i=1}^{n} x^\mu_{i,i+1}\ln\Big(\frac{x^2_{i,i+2}}{x^2_{i-1,i+1}}\Big)$$

The same relations also hold at strong coupling [Alday,Maldacena],[Komargodski]

Symmetries of all-loop amplitudes

- ✔ IR divergences preserve supersymmetry but break conformal + dual conformal symmetry
- ✔ Symmetries ($p, q, \bar{q}, P, Q, \bar{S}, \ldots$) survive loop corrections, other ($s, \bar{s}, k, K, S, \bar{Q}, \ldots$) are broken



- ✔ Dual conformal K-anomaly is *universal* for all amplitudes (MHV, NMHV,...) [Drummond,Henn,GK,Sokatchev]

$$K^{\alpha\dot{\alpha}}\, A_n \equiv \sum_{i=1}^{n} \left[2x_i^{\alpha\dot{\alpha}}(x_i \cdot \partial_{x_i}) - x_i^2 \partial_{x_i}^{\alpha\dot{\alpha}} \right] A_n = \frac{1}{2}\Gamma_{\rm cusp}(g^2) \sum_{i=1}^{n} x_{i,i+1}^{\alpha\dot{\alpha}} \ln\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} \right) A_n$$

- ✔ The $s-$ and $\bar{Q}-$anomalies are hard to control

Dual conformal anomaly at work

Consequences of the dual conformal K-anomaly for the *finite* part of MHV amplitude:

✔ $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x_{i,i+1}^2 = 0$)

$\Longrightarrow$ the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$\ln A_4^{\mathrm{MHV}} = \frac{1}{4}\Gamma_{\mathrm{cusp}}(g^2)\ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{ const },$$

$$\ln A_5^{\mathrm{MHV}} = -\frac{1}{8}\Gamma_{\mathrm{cusp}}(g^2)\sum_{i=1}^{5}\ln\left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right)\ln\left(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2}\right) + \text{ const}$$

✔ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios $u_{ijkl} = \frac{x_{il}^2 x_{jk}^2}{x_{ik}^2 x_{jl}^2}$

General solution to the Ward identity contains remainder function of the conformal cross-ratios

✔ State-of-art calculation : remainder functon for $n = 6$ MHV amplitude

✗ $A_6 = [\text{BDS ansatz}] \times [\text{Remainder func.}]$ [Drummond,Henn,GK,Sokatchev],[Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

✗ Analytical expression at weak coupling [Del Duca, Durer,Smirnov],[Zhang],[Goncharov,Spradlin,Vergu,Volovich]

✗ Strong coupling prediction [Alday, Gaiotto,Maldacena]

✗ Rich structure at strong coupling (integrability, Y-system, TBA) [Alday, Gaiotto,Maldacena,Sever,Viera]

Correlation functions

- Protected superconformal operators made from scalars $\phi_{AB} = \frac{1}{2}\epsilon_{ABCD}\bar{\phi}^{CD}$

$$\mathcal{O}(x) = \mathrm{Tr}(\phi_{12}\phi_{12})\,, \qquad \tilde{\mathcal{O}}(x) = \mathrm{Tr}(\bar{\phi}^{12}\bar{\phi}^{12})$$

  Scaling dimensions do not receive quantum corrections

- Simplest correlation function

$$G_4 = \langle \mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\mathcal{O}(x_3)\tilde{\mathcal{O}}(x_4)\rangle = \frac{N_c^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}\mathcal{F}(u,v;\lambda)$$

  Conformal cross-ratios

$$u = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

  Conformal invariant (coupling dependent) function $\mathcal{F}(u,v;\lambda)$ is finite as long as $x_i \neq x_j$

  The limit $x_i \to x_j$ corresponds to the standard OPE

- New limit: let *all the neighboring points* be light-like separated at the same time

$$x_{i,i+1}^2 \;\to\; 0\,, \qquad x_i \neq x_{i+1}\,, \qquad (i = 1,\ldots,n)$$

  Conformal cross-ratios vanish $u,\ v \to 0$

Correlation functions on the light-cone

The light-cone limit of G_4 is singular:

(i) For $x^2_{i,i+1} \to 0$ the correlator develops pole singularities already at tree level

$$G_4^{(\text{tree})} \sim \frac{N_c^2}{x^2_{12}x^2_{23}x^2_{34}x^2_{41}} + \text{subleading terms}$$

The way out – consider the ratio

$$\mathcal{F}_4 \equiv \lim_{x^2_{i,i+1}\to 0} G_4(x_i)/G_4^{(\text{tree})}(x_i)$$

(ii) In addition, loop integrals develop additional light-cone singularities (cross-ratios vanish $u, v \to 0$)

$$\mathcal{F}_4 = 1 + a\,\frac{i}{\pi^2}\int \frac{d^4x_0\,x^2_{13}x^2_{24}}{x^2_{10}x^2_{20}x^2_{30}x^2_{40}} + \ldots = 1 - a\ln u \ln v + \ldots\,.$$

Leading divergent terms can be resummed *to all loops*

$$\mathcal{F}_4 \sim \exp\left(-\frac{1}{2}\Gamma_{\text{cusp}}(a)\ln u \ln v\right)$$

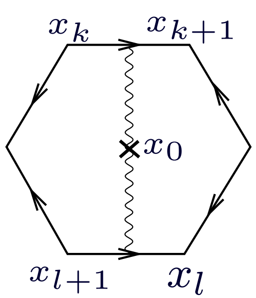
✔ Almost like 1-loop correction to the amplitude $A_4(p_i)$ except that $p_i = x_{i,i+1}$ should be light-like,

$x^2_{i,i+1} = p_i^2 = 0 \quad \Longrightarrow \quad$ *light-cone singularities need to be regularized*

Correlation functions on the light-cone II

- Two possible choices of a regularization procedure:
  - Use the small distances $x^2_{i,i+1}$ as a cutoff in $D = 4$ dimensions;
  - Employ dimensional regularization with $D = 4 - 2\epsilon$ and set $x^2_{i,i+1} = 0$ from the start
- One-loop calculation of the correlation function in the dimensional regularization for $x^2_{i,i+1} = 0$

$$\frac{d}{da}\left[G_n/G_n^{(\mathrm{tree})}\right]_{\mathrm{l.c.}} =$$



$$\sim \sum_{k>l=1}^{n} \int d^D x_0 \, T^{\mu\nu}(x_k, x_0, x_{k+1}) \, T_{\mu\nu}(x_l, x_0, x_{l+1})$$

- Result of explicit calculation ($x_{s_k} = x_k - s_k x_{k,k+1}$)

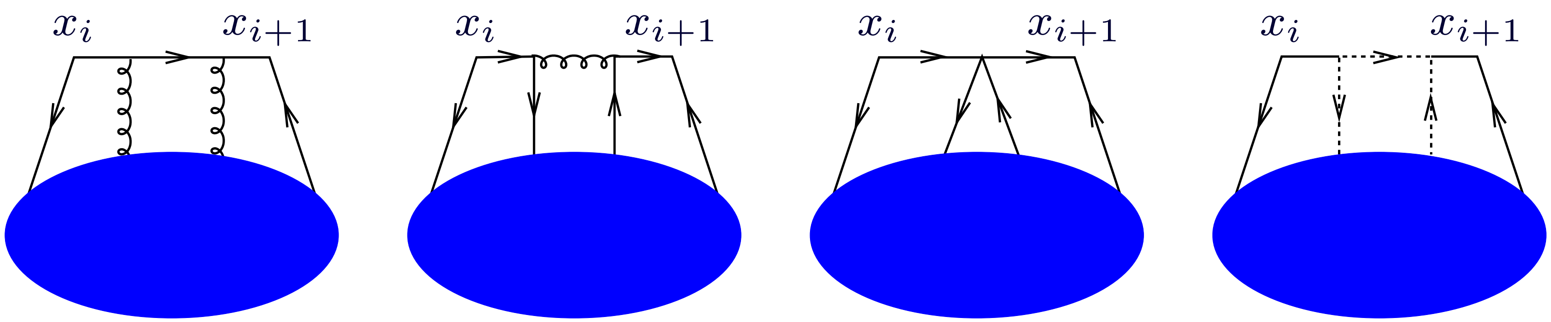
$$\ln\left[G_n/G_n^{(\mathrm{tree})}\right]_{\mathrm{l.c.}} = (ig)^2 N_c \sum_{k>l} \int_0^1 ds_k \int_0^1 ds_l \, x^\mu_{k,k+1} x^\nu_{l,l+1} D_{\mu\nu}(x_{s_k,s_l}) + \ldots = 2 \ln W[C_n]$$

Coincides with the one-loop expression for the light-like polygon Wilson loop!

$$\left[G_n/G_n^{(\mathrm{tree})}\right]_{\mathrm{l.c.}} \propto (W[C_n])^2$$

The normalization factor depends on the regularization employed

- Different types of diagrams contributing to the correlation function $G_n = \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle$



(a) (b) (c) (d)

Solid, wavy and dashed lines denote scalars, gluons and gluinos, respectively

The shadowed blobs stand for the rest of the diagram involving the remaining operators.

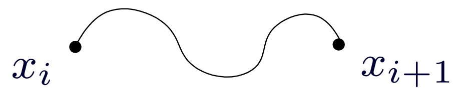
- For $x^2_{i,i+1} \to 0$ *the leading contribution* comes from diagrams (a) only

$$G_n \to \langle 0 | \mathrm{Tr}[S(x_1, x_2; A) S(x_2, x_3; A) \dots S(x_n, x_1; A)] | 0 \rangle$$

$S(x_i, x_{i+1}; A) =$ propagator of scalar in a background gauge field

- First-quantied description (random walk of a scalar particle)

$$S(x_i, x_{i+1}; A) = \sum_C \mathrm{e}^{-iL(C)}\, \mathrm{P}\, \mathrm{e}^{i \int_C dx^\mu A_\mu(x)}\,, \qquad C =$$

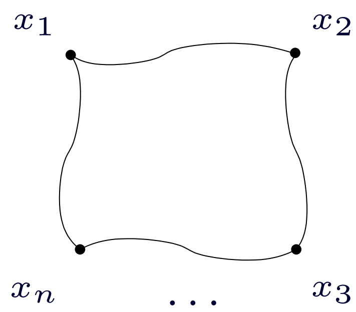


Interaction with gauge field $\to$ path-ordered exponential in the *adjoint* of the $SU(N_c)$

From correlation functions to Wilson loops II

- Correlation function on the light-cone

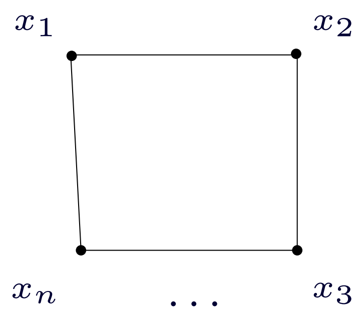
$$G_n \to \sum_C \mathrm{e}^{-iL(C)} \langle 0 | \mathrm{Tr}_{\mathrm{adj}}\, \mathrm{P}\, \mathrm{e}^{i \oint_C dx^\mu A_\mu(x)} | 0 \rangle \,, \qquad C =$$



Infinitely fast particle interacting with a slowly varying gauge field (for $x^2_{i.i+1}\mu^2 \ll 1$ only!)

- The path-integral is dominated by the saddle point $C_n =$ classical trajectory of a particle

$$G_n \to G_n^{(\mathrm{tree})} \times \langle 0 | \mathrm{Tr}_{\mathrm{adj}}\, \mathrm{P}\, \mathrm{e}^{i \oint_{C_n} dx^\mu A_\mu(x)} | 0 \rangle \,, \qquad C_n =$$



- All-loop result, valid in any gauge theory

$$\lim_{x^2_{i,i+1} \to 0} \left(G_n / G_n^{(\mathrm{tree})} \right) = W_{\mathrm{adj}}[C_n] = \left(W[C_n] \right)^2 + O(1/N_c^2)$$

$W[C_n]$ satisfies anomalous conformal Ward identity $\longrightarrow$ new results for light-cone asymptotics of the correlation functions G_n!

From duality to triality

Wilson loops/Scattering amplitudes duality

$$\ln\left(A_n^{\mathrm{MHV}}/A_n^{(\mathrm{tree})}\right) = \ln\left(W[C_n]\right) + O(1/N_c^2)$$

Correlation functions/Wilson loops duality

$$\lim_{x_{i,i+1}^2\to 0} \ln\left(G_n/G_n^{(\mathrm{tree})}\right) = 2\ln\left(W[C_n]\right) + O(1/N_c^2)$$

... leading to Correlation functions/Scattering amplitudes duality

$$\lim_{x_{i,i+1}^2\to 0} \ln\left(G_n/G_n^{(\mathrm{tree})}\right) = 2\ln\left(A_n^{\mathrm{MHV}}/A_n^{(\mathrm{tree})}\right) + O(1/N_c^2)$$

- ✔ The planarity condition is realized differently for correlators and scattering amplitudes:
  - ✗ For A_n the planar diagrams have the topology of a disk
  - ✗ For G_n the planar diagrams have the topology of a sphere

  The duality implies nontrivial relations between two different class of multi-loop integrals
- ✔ Has been verified to two loops for $n = 4,\ 5$
- ✔ The duality $G_n \leftrightarrow A_n^{\mathrm{MHV}}$ can be established directly, without invoking Wilson loops

From $\mathcal{N}=4$ SYM to QCD

- ✔ Scattering amplitudes in QCD are much more complicated

  $\mathcal{A}_4(s,t)$ to two loops = [4 pages mess]

  no relation to Wilson loops for generic s and t ... but let us examine the Regge limit $s \gg -t$

- ✔ Planar four-gluon QCD scattering amplitude in the Regge limit $s \gg -t$ [Schnitzer'76],[Fadin,Kuraev,Lipatov'76]

$$\mathcal{M}_4^{(\mathrm{QCD})}(s,t) \sim (s/(-t))^{\omega(-t)} + \dots$$

- ✔ Rectangular Wilson loop in QCD in the Regge limit $|x_{13}^2| \gg |x_{24}^2|$

$$W^{(\mathrm{QCD})}(C_4) \sim \left(x_{13}^2/(-x_{24}^2)\right)^{\omega(-x_{24}^2)} + \dots$$

  The scattering amplitude/Wilson loop duality relation holds in QCD in the Regge limit only [GK'96]

$$\ln \mathcal{M}_4^{(\mathrm{QCD})}(s,t) = \ln W^{(\mathrm{QCD})}(C_4) + O(t/s)$$

  while in $\mathcal{N}=4$ SYM it is exact for arbitrary t/s!

- ✔ Dual conformal symmetry is present in the BFKL equation [Gomez,Sabio Vera],[Prygarin]

Recent developments and open questions

- ✔ Classification of all invariants of conformal + dual conformal symmetry (Grassmanians, Integral Geometry, ...) [Arkani-Hamed et al],[Mason,Skinner],[Drummond,Ferro],[GK,Sokatchev]
- ✔ New evidences for dual conformal symmetry of non-MHV amplitudes (DHKS conjecture) : $n = 6$ NMHV amplitude to two loops [Kosower,Roiban,Vergu]
- ✔ New technical tools in computing amplitudes: IR regularization by giving vev's to scalar (Coulomb branch of $\mathcal{N} = 4$ SYM) [Alday,Henn,Plefka,Schuster]
- ✔ New approaches to constructing four-dimensional *integrands* for scattering amplitudes
  - ✗ Correlation functions with Lagrangian insertions [Eden,GK,Sokatchev]
  - ✗ Recursive BCFW construction in momentum twistor space [Arkani-Hamed et al],[Boels]

Open problems:

- ✔ Extend the dualities to non-MHV amplitudes
- ✔ Identify a dual object analogous to the light-like Wilson loop for generic non-MHV amplitude in planar $\mathcal{N} = 4$ SYM theory
- ✔ To understand *why* the duality works. Are there some hidden symmetries of the correlator, which fix it to a unique form?
- ✔ Scattering amplitudes seem to be integrable. What are the underlying integrable structures?

Conclusions

Maximally supersymmetric Yang-Mills theory is “harmonic oscillator of 21st century”:

- ✔ An excellent testing ground for computing QCD scattering amplitudes needed for precise theoretical predictions at hadron colliders

- ✔ Unexpected interconnections (Wilson loos/Correlators/Amplitude duality) and hidden symmetries (dual symmetry)

- ✔ Dual symmetry is also present in QCD but in the Regge limit only ... yet another glimpse of QCD/string duality?!