Conference "Gauge theories and the structure of spacetime"

Kolymbari, Crete, September 17, 2010

Integrability in N=4 SYM theory

Vladimir Kazakov

(ENS, Paris)

Integrability in AdS/CFT

- Integrability is usually a 2D phenomenon (statistical mechanics on 2D lattice, quantum spin chains, (1+1)D quantum field theories, etc)
- Now possible for some super-conformal -Yang-Mills theories at D=3,4…
- Based on AdS/CFT duality to a very special 2D QFT's - superstring σ-models
- In light cone gauge it looks like a massive σ-model. Most of 2D integrability tools then applicable: S-matrix, asymptotic Bethe ansatz (ABA), Thermodynamic Bethe Ansatz (TBA) for finite volume spectrum, etc.
- .... Y-system (for planar AdS_5/CFT_4 , AdS_4/CFT_3 ,...)

  Conjecture: it calculates anomalous dimensions of all local operators in four-dimensional planar superconformal Yang-Mills (SYM) with N=4 supersymmetries, at any coupling

  Gromov,V.K.,Vieira

- Further simplification: Y-system as Hirota discrete integrable dynamics

Examples of Integrability in Gauge Theories

- QCD in Regge limit

Lipatov
Faddeev, Korchemsky

- Spectrum of anomalous dimensions in planar N=4 SYM (AdS/CFT, TBA, Y-system)

- n-point correlators in N=4 SYM (talks of R.Janik, M.Costa, A.Tseytlin)

- Amplitudes and Wilson loops (Yangian symmetry, Y-system for strong coupling,…) (G.Korchemsky's talk)

- New integrable dualities
- ABJM: $AdS_4 \times CP^3$
- $AdS_3 \times S^3 \times T^4$
- N=2 SQCD, $N_f = 2N_c$

Minahan-Zarembo
Gromov,Vieira,
Gromov,V.K.,Vieira

Zarembo

Gadde,Pomoni,Rastelli

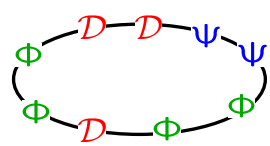
- Different from BPS-integrability (starting from Seiberg-Witten …) : in N=4 SYM we sum up 4D planar Feynman graphs!

N=4 SYM as a superconformal 4D QFT

$$\mathcal{S}_{SYM} = \frac{1}{\lambda}\int d^4x\, \mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + \bar{\Psi}\mathcal{D}\Psi + \bar{\Psi}\Phi\Psi + [\Phi,\Phi]^2\right)$$

- Operators in 4D

$$\mathcal{O}(x) = \mathrm{Tr}\,[\underbrace{\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots}_{L\,=\,\#\mathrm{bosons}\,+\,\#\mathrm{fermions}}]\,(x)$$



- 4D Correlators (superconformal!):

non-trivial functions
of 'tHooft coupling λ!

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \frac{\delta_{ij}}{|x|^{2\Delta_j(\lambda)}}$$

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}(\lambda)}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}|x_{31}|^{\Delta_i+\Delta_k-\Delta_j}}$$

*

Anomalous dimensions in various limits

$$\delta(\lambda, L, \ldots) = \Delta - \Delta_0$$

- Perturbation theory: $\lambda \to 0\,, \quad L$ – fixed

  $$\delta \simeq \lambda f_1(L) + \lambda^2 f_2(L) + \lambda^3 f_3(L) \ldots$$

  1,2,3…-loops: integrable spin chain

  Minahan,Zarembo
  Beisert,Kristijanssen,Staudacher

- BFKL approx. for twist-2 operators $\mathcal{O}_S = \mathrm{Tr}\left[Z D^S Z\right]$

  $$\frac{S+1}{-\lambda} = \Psi\left(\frac{1}{2} - \frac{\Delta}{2}\right) + \Psi\left(\frac{1}{2} + \frac{\Delta}{2}\right) - 2\Psi(1)$$

  $\lambda \to 0\,, \quad S+1 \to 0\,, \quad \frac{S+1}{\lambda}$ – fixed

  Kotikov, Lipatov

- String (quasi)-classics: $\lambda \to \infty\,, \quad L \to \infty\,, \quad \frac{\lambda^{1/2}}{L}$ – fixed

  $$\delta \simeq \sqrt{\lambda}\left[g_1\left(\frac{\sqrt{\lambda}}{L}\right) + \frac{1}{\sqrt{\lambda}}\, g_2\left(\frac{\sqrt{\lambda}}{L}\right) + \frac{1}{\lambda}\, g_3\left(\frac{\sqrt{\lambda}}{L}\right) + \ldots\right]$$

  Finite gap method, Bohr-Sommerfeld

  Frolov,Tseytlin,
  V.K.,Marshakov,Minahan,Zarembo
  Beisert,V.K.,Sakai,Zarembo
  Roiban,Tseytlin,
  Gromov,Vieira

- Long operators, no wrappings: λ – fixed, $\quad L \to \infty$

  $$\delta \simeq L\, h(\lambda) + \mathcal{O}\left(e^{-L\,\epsilon(\lambda)}\right)$$

  Asymptotic Bethe Ansatz (ABA)

  Beisert,Staudacher
  Beisert,Eden,Staudacher

- Strong coupling, short operators: $\lambda \to \infty\,, \quad L$ – fixed

  $$\delta \simeq \lambda^{1/4} r_1(L) + \lambda^{-1/4} r_2(L) + \mathcal{O}\left(\lambda^{-3/4}\right)$$

  Worldsheet perturbation theory

  Gubser,Klebanov,Polyakov

- Exact dimensions (all wrappings): Y-system and TBA

  Gromov,V.K.,Vieira

  Gromov,V.K.,Vieira
  Bombardelli,Fioravanti,Tateo
  Gromov,V.K.,Kozak,Vieira
  Arutyunov,Frolov

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SYM is Dual to Supersting σ-model on $AdS_5 \times S^5$

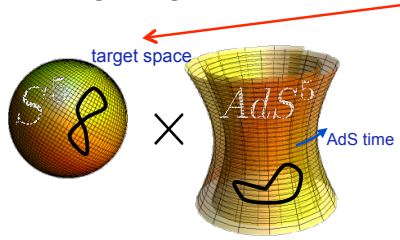
Super-conformal N=4 SYM symmetry PSU(2,2|4) → isometry of string target space

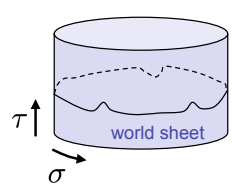
- 2D σ-model on a coset

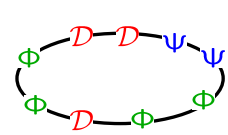
  $$\frac{\mathrm{PSU}(2,2\,|\,4)}{\underbrace{\mathrm{SO}(1,4) \times \mathrm{SO}(5)}} \qquad g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathrm{SU}(2,2|4)$$

  fermions

  $$J = -g^{-1} dg = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)} \in su(2,2|4)$$







- Metsaev-Tseytlin action

  $$S_{MT} = \sqrt{\lambda}\ \mathrm{str} \int_{\mathcal{M}_2} \left[\left(J^{(2)}\right)^2 - J^{(1)} \wedge J^{(3)}\right]$$

Dimension of YM operator $\Delta_A(\lambda)$ = Energy of a string state

Classical integrability of superstring on $AdS_5 \times S^5$

- String equations of motion and constraints can be recasted into zero curvature condition

$$(d + \mathcal{A}(z)) \wedge (d + \mathcal{A}(z)) = 0,$$

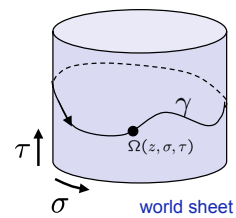
Mikhailov,Zakharov
Bena,Roiban,Polchinski

for Lax connection ***rationally*** depending on a spectral parameter z

$$\mathcal{A}(z) = J^{(0)} + \frac{1}{2}\left(z^{-2} + z^2\right) J^{(2)} + \frac{1}{2}\left(z^{-2} - z^2\right) * J^{(2)} + z^{-1} J^{(1)} + z\, J^{(3)}$$

- Monodromy matrix $\Omega(z) = P \exp \oint_\gamma \mathcal{A}(z)$

encodes infinitely many conservation lows



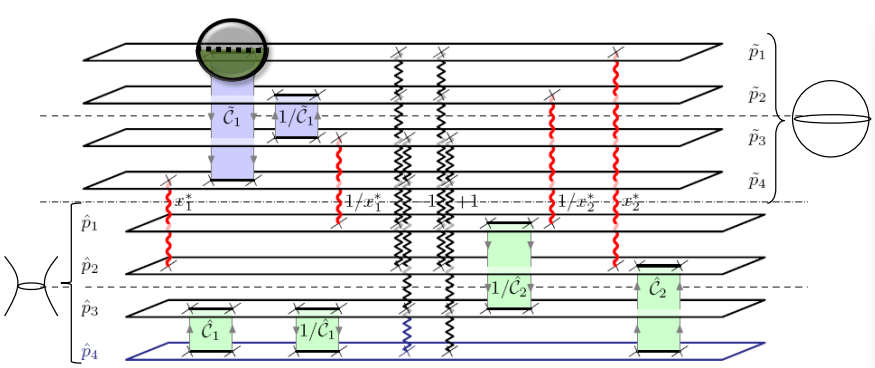
- Finite gap method can be used to construct particular classical solutions

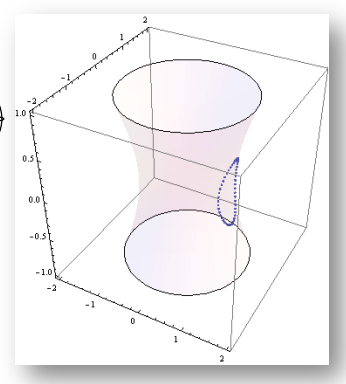
Its,Matveev,Dubrovin,Novikov,Krichever

Finite gap method for classical superstring

- Algebraic curve for quasi-momenta: $\mathcal{P}(p,z) = \text{sdet}\left(e^{ip} - \Omega(z)\right) \sim \frac{0}{0}$

$$\Omega(z) = \{e^{i\tilde{p}_1(z)}, e^{i\tilde{p}_2(z)}, e^{i\tilde{p}_3(z)}, e^{i\tilde{p}_4(z)} || e^{i\hat{p}_1(z)}, e^{i\hat{p}_2(z)}, e^{i\hat{p}_3(z)}, e^{i\hat{p}_4(z)}\}$$





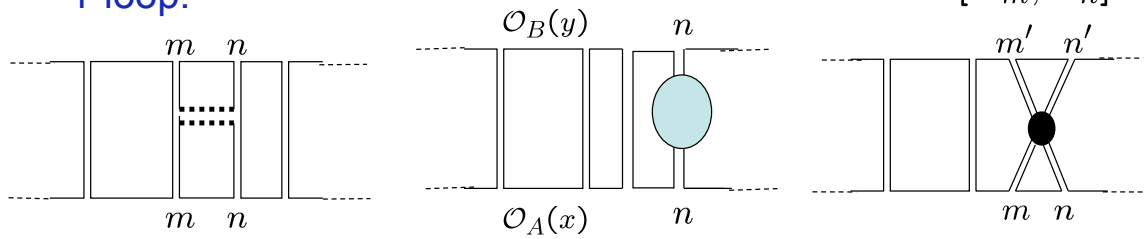
- Cuts (gaps) connecting Riemann sheets define the motion
- Quantization: Cuts consist of condensed Bethe roots carrying the quanta of excitations.

V.K.,Marshakov,Minahan,Zarembo
Beisert,V.K.,Sakai,Zarembo
Gromov,Vieira

Weak coupling calculation by point splitting

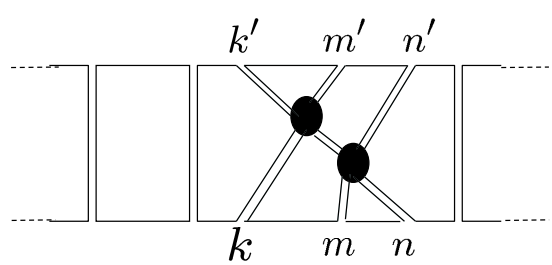
- Tree level: $\Delta_0 = L$ (degeneracy)
- 1-loop:

$\mathrm{Tr}\,[\Phi_m, \Phi_n]^2$



nontrivial action on R-indices.

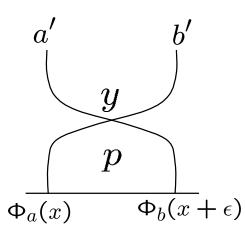
- 2-loop:



Perturbative integrability

- Interaction:

$$\frac{1}{2}\mathrm{Tr}\,[\Phi_a, \Phi_b]^2 = \mathrm{Tr}\,(\Phi_a\Phi_b)^2 - \mathrm{Tr}\,\Phi_a^2\Phi_b^2$$



$$= \frac{\lambda}{8\pi^2}\log\frac{1}{\epsilon}\,(\delta_{ab'}\delta_{ba'} - 2\delta_{aa'}\delta_{bb'} + C\delta_{aa'}\delta_{bb'})$$

$$X = \Phi_1 + i\Phi_2$$
$$Y = \Phi_3 + i\Phi_4$$
$$Z = \Phi_5 + i\Phi_6$$

- "Vacuum" – BPS operator $\mathcal{O}_{\mathrm{BPS}} = \mathrm{Tr}\ Z^L$
- Example: SU(2) sector $\mathcal{O}_{SU(2)} = \mathrm{Tr}\ X^J Z^{L-J}$ + permutations
- Dilatation operator = Heisenberg Hamiltonian, integrable by Bethe ansatz!

Minahan,Zarembo
Beisert,Kristijanssen,Staudacher

$$\hat{D} = L + \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(1 - \sigma_l\cdot\sigma_{l+1}\right) +$$

$$\left(\frac{\lambda}{16\pi^2}\right)^2\sum_{l=1}^{L}\left(\left(1 - \sigma_l\cdot\sigma_{l+2}\right) - 4\left(1 - \sigma_l\cdot\sigma_{l+1}\right)\right) + O(\lambda^3)$$

$$\Delta = \Delta^{(0)} + \lambda\Delta^{(2)} + \lambda^2\Delta^{(4)} + \ldots$$

Exact spectrum at one loop

$$\hat{D} = L + \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(1-\sigma_l\cdot\sigma_{l+1}\right) + O(\lambda^2)$$

$$-Z-Z-X-Z-Z-X-Z-$$

Rapidity parametrization: $\quad e^{ip} = \dfrac{u+i/2}{u-i/2}$

$$\boxed{\left(\frac{u_k+i/2}{u_k-i/2}\right)^L = \prod_{(k\neq)j=1}^{J}\frac{u_k-u_j+i}{u_k-u_j-i}}$$

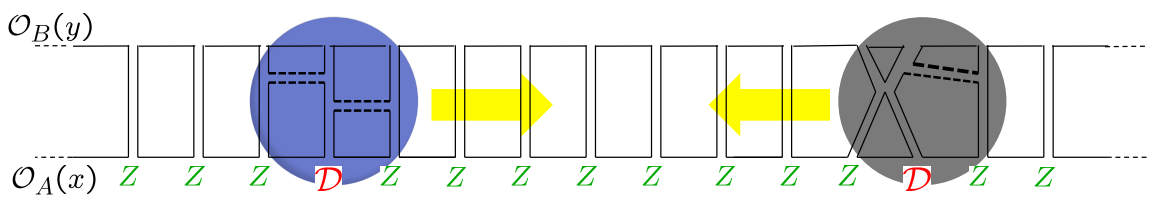
Bethe'31

Anomalous dimension:

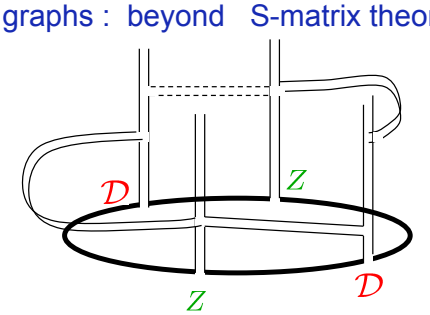
$$\Delta - L = \frac{\lambda}{8\pi^2}\sum_{k=1}^{J}\frac{1}{u_k^2+1/4}$$

SYM perturbation theory and finite size (wrapping) effects

- Unwrapped graphs - described by asymptotic scattering on 1D "spin chain"



- Wrapped graphs : beyond S-matrix theory



$$\mathcal{O}_{Konishi} = \mathrm{Tr}\ [\mathcal{D}, Z]^2$$

- We need to take into account the finite size effects - Y-system needed
- Important observation on string side: Lüscher corrections + integrability reproduce the direct perturbation theory at 4 loops.

Bajnok,Janik

Exact one-particle dispersion relation

Santambrogio,Zanon
Beisert,Dippel,Staudacher
N.Dorey

- Exact one particle dispersion relation:
- Bound states (fusion!)

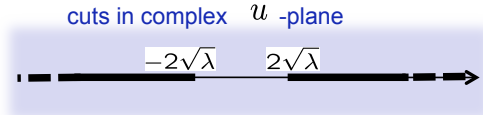
$$\epsilon_a(p) = \sqrt{a^2 + \lambda \sin^2 \frac{pa}{2}}$$

$$-\mathcal{Z}-\mathcal{Z}-\underset{u-\frac{i}{2}}{\mathcal{D}}-\mathcal{Z}-\mathcal{Z}-\underset{u+\frac{i}{2}}{\mathcal{D}}-\mathcal{Z}-$$

- Cassical spectral parameter z related to quantum one by Zhukovsky map

$$u = \sqrt{\lambda}\left(z + \frac{1}{z}\right), \qquad z = \frac{1}{2\sqrt{\lambda}}\left(u + \sqrt{4\lambda - u^2}\right)$$

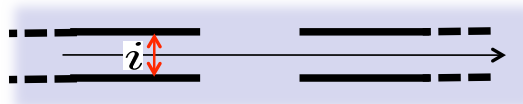
cuts in complex u -plane



- Parametrization for the dispersion relation:

$$p(u) = \frac{1}{i}\log\frac{z(u+i/2)}{z(u-i/2)}$$

$$\epsilon(u) = 2i\sqrt{\lambda}z(u-i/2) - 2i\sqrt{\lambda}z(u+i/2) + 1$$

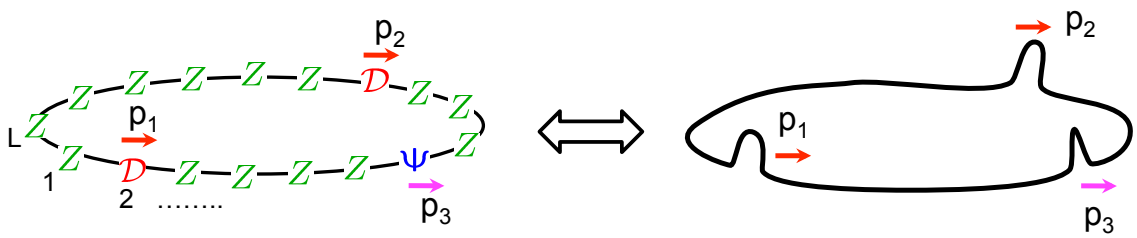


- Motivation: differential $p(u)du$ defines the canonical simplectic structure on the algebraic curve: Bohr-Sommerfeld quantization of a finite gap solution

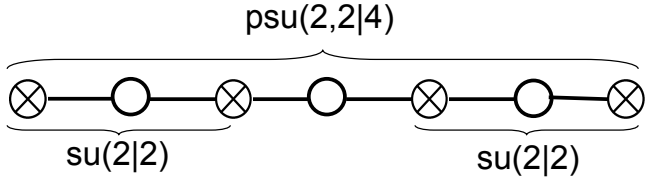
$$\oint_{A_j} p(u)\, du = 2\pi(n_j + \frac{1}{2})$$

N.Dorey, Vicedo

S-matrix and integrable superspin chain



- Light cone gauge breaks the global and world-sheet Lorentz symmetries :



- S-matrix of AdS/CFT via bootstrap à-la A.&Al.Zamolodchikov

$$\hat{S}_{PSU(2,2|4)}(p_1,p_2) = S_0^2(p_1,p_2) \times \hat{S}_{SU(2|2)}(p_1,p_2) \times \hat{S}_{SU(2|2)}(p_1,p_2)$$

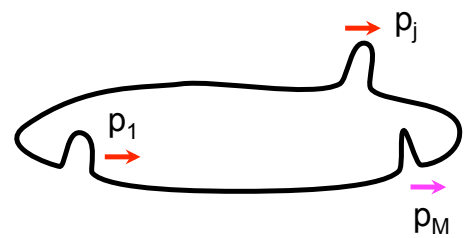
Staudacher
Beisert
Janik

Shastry's R-matrix
of Hubbard model

Asymptotic Bethe Ansatz (ABA)

Beisert,Eden,Staudacher

$$0 = \left(e^{iLp_j} \prod_{k \neq j}^{M} \hat{S}(p_j, p_k) - 1 \right) \Bigg| \quad \Bigg\rangle$$



- This periodicity condition is diagonalized by nested Bethe ansatz

- Energy of state

$$E = \Delta - \Delta_0 = \sum_{j=1}^{M} \epsilon(p_j) \quad + \quad O\left(e^{-c\,L}\right)$$

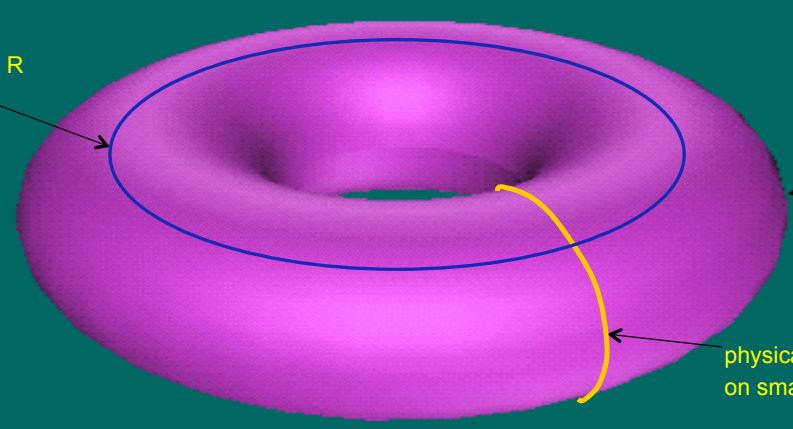
finite size corrections,
important for short operators!

- Results: ABA for dimensions of long YM operators (e.g., cusp dimension).

TBA for finite size (Al.Zamolodchikov trick)

Gromov,V.K.,Vieira
Bombardelli,Fioravanti,Tateo
Gromov,V.K.,Kozak,Vieira
Arutyunov,Frolov

Ambjorn,Janik,Kristjansen
Arutyunov,Frolov

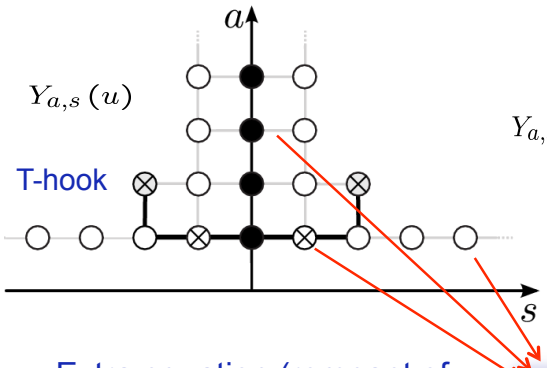


$$\mathcal{Z} = \sum_n e^{-E_n^{mir}(R)L} = \sum_n e^{-E_n^{ph}(L)R} \xrightarrow[R\to\infty]{} e^{-E_0^{ph}(L)R}$$

- Large R : mirror momenta localize on poles of S-matrix $\rightarrow$ bound states

$$S\left(p(u), p(v)\right) \sim \frac{1}{u - v + i}$$

Y-system for excited states of AdS/CFT at finite size

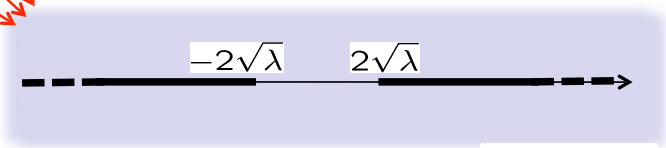


Gromov,V.K.,Vieira

$$Y_{a,s}\left(u+\frac{i}{2}\right)Y_{a,s}\left(u-\frac{i}{2}\right)=\frac{\left[1+Y_{a,s+1}\right]\left[1+Y_{a,s-1}\right]}{\left[1+\frac{1}{Y_{a+1,s}}\right]\left[1+\frac{1}{Y_{a+1,s}}\right]}$$

- Complicated analyticity structure in u dictated by non-relativistic dispersion

cuts in complex u-plane



- Extra equation (remnant of classical $z \leftrightarrow \frac{1}{z}$ monodromy):

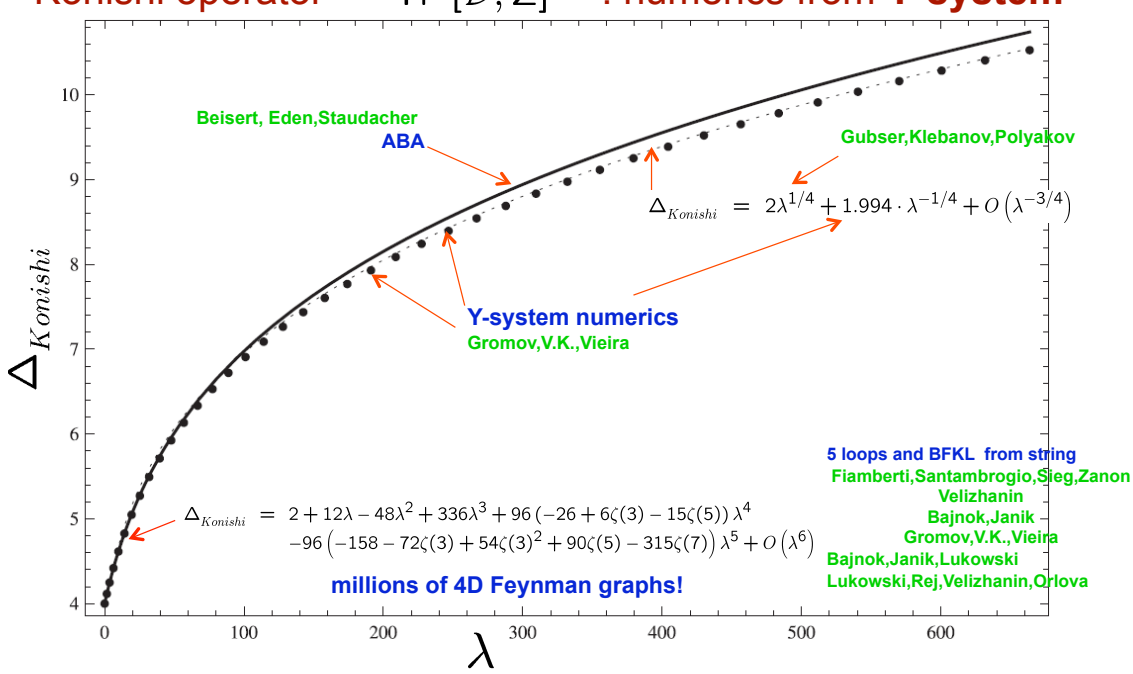
$$Y_{2,\pm 2}(u+i0)\cdot Y_{1,\pm 1}(u-i0)=1$$

- Energy : (anomalous dimension) $\Delta-\Delta_0=f_{min}=\int\frac{du}{2\pi i}\,\partial_u\tilde{\epsilon}_a\,\log\left(1+Y_{a,0}\right)$
- u_j obey the exact Bethe eq.: $Y_{1,0}(u_j)+1=0$
- Knowing analyticity one transforms functional Y-system into integral (TBA):

$$\log Y_{a,s}=L\,\delta_{a,0}\frac{\partial}{\partial u}\epsilon_a^{mir}+\sum_{a',s'}K_{a,s;a',s'}*\log\left(1+Y_{a',s'}(u)\right)$$

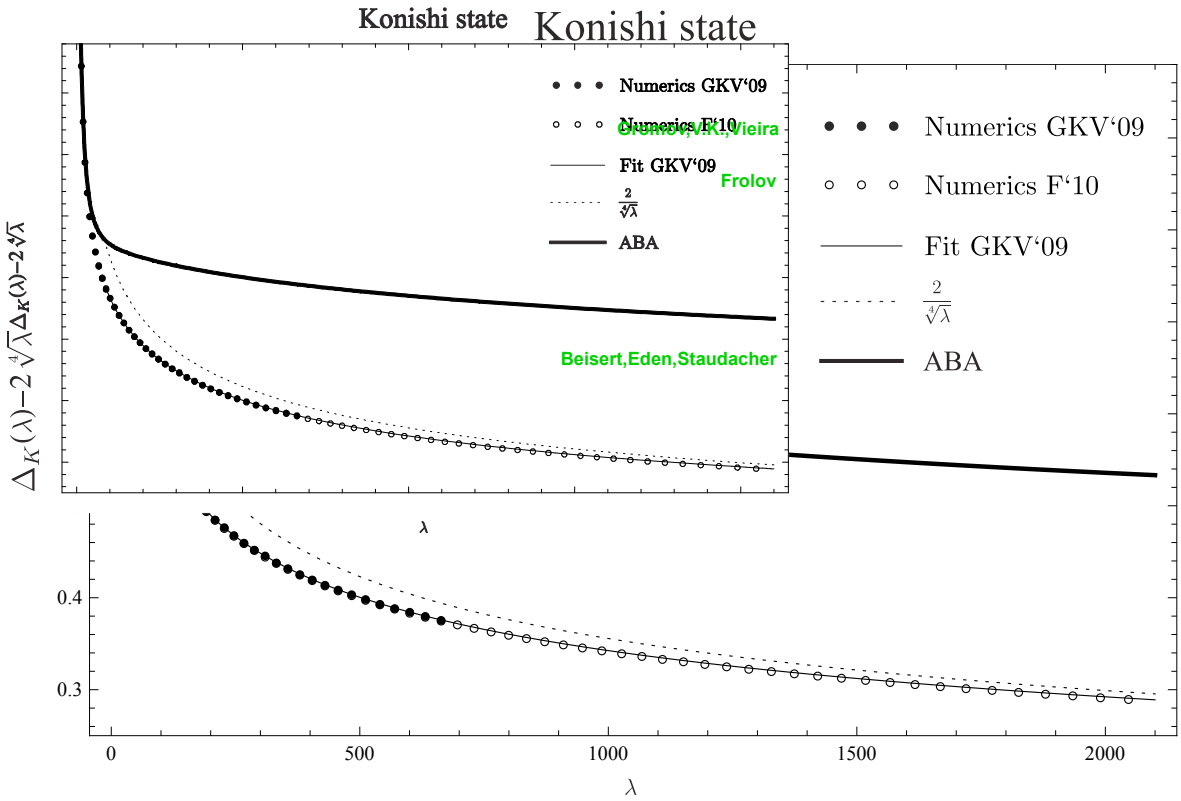
Gromov,V.K.,Vieira
Bombardelli,Fioravanti,Tateo
Gromov,V.K.,Kozak,Vieira
Arutyunov,Frolov
Cavaglia, Fioravanti, Tateo

Konishi operator $\mathrm{Tr}\,[\mathcal{D},Z]^2$: numerics from **Y-system**



- Y-system passes all known tests

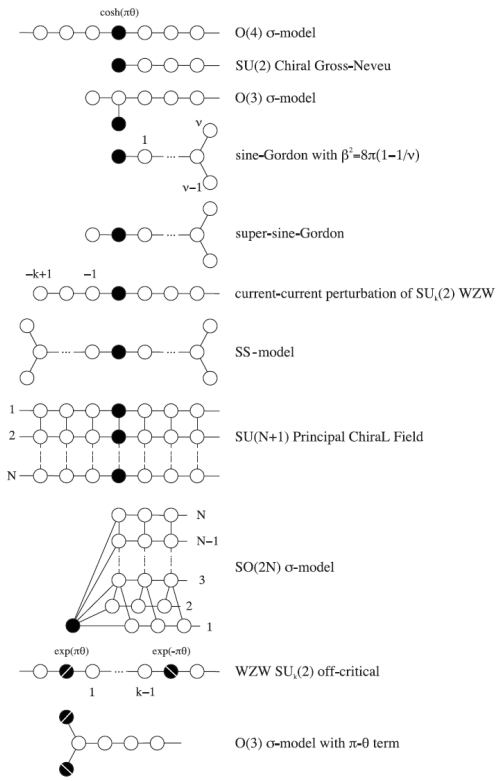
Konishi operator $\mathrm{Tr}\,[\mathcal{D}, Z]^2$: numerics from **Y-system**

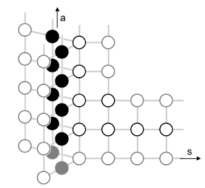


Y-system looks very "simple" and universal!

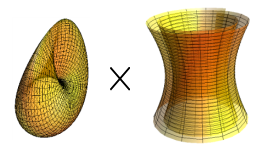
- Similar systems of equations in all known integrable σ-models
- What are its origins? Could we guess it without TBA?

Y-systems for other σ-models





New AdS/CFT's : new Y-systems
3d ABJM model: $CP^3 \times AdS_4$, …



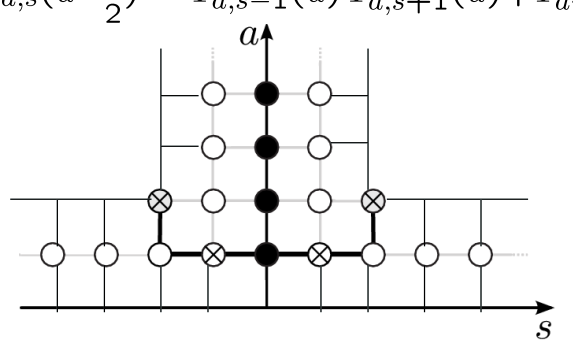
Gromov,V.K.,Vieira
Bombardelli,Fiorvanti,Tateo
Gromov,Levkovich-Maslyuk

Y-system and Hirota eq.: discrete integrable dynamics

- Relation of Y-system to T-system (Hirota equation) (the Master Equation of Integrability!)

$$Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$$

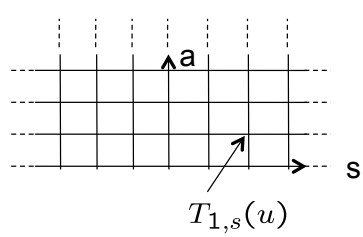
$$T_{a,s}(u+\tfrac{i}{2})T_{a,s}(u-\tfrac{i}{2}) = T_{a,s-1}(u)\,T_{a,s+1}(u)+T_{a+1,s}(u)\,T_{a-1,s}(u)$$



For spin chains :
Klumper,Pearce
Kuniba,Nakanishi,Suzuki
For QFT's:
Al.Zamolodchikov
Bazhanov,Lukyanov,A.Zamolodchikov

- Discrete classical integrable dynamics!
- General solution in half-plane lattice:

$$T_{a,s}(u) = \mathrm{Det}_{1\le k,j\le a}\; T_{1,s+k-j}\left(u+i\frac{k+j}{2}\right)$$



For spin chains :
Bazhanov,Reshetikhin
Cherednik
V.K.,Vieira (for the proof)

Quasiclassical solution of AdS/CFT Y-system

- Classical limit: highly excited long strings/operators, strong coupling:

$$L \sim \sqrt{\lambda} \sim u \to \infty$$

- Explicit u-shift in Hirota eq. dropped (only slow parametric dependence)

$$T_{a,s}\left(u+\cancel{\frac{i}{2\sqrt{\lambda}}}\right)T_{a,s}\left(u-\cancel{\frac{i}{2\sqrt{\lambda}}}\right)=T_{a,s-1}(u)\,T_{a,s+1}(u)+T_{a+1,s}(u)T_{a-1,s}(u)$$

- (Quasi)classical solution - su(2,2|4) character of classical monodromy matrix!

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$$T_{a,s}=\mathrm{Tr}_{a,s}\,\Omega$$

- (Quasi-)classics restored entirely from the Y-system!

Classical finite gap solution

- Can be restored from Y-system in the limit $L \sim \sqrt{\lambda} \sim u \to \infty$

$$\begin{aligned}
y_{\hat{1}} &= e^{-i\hat{p}_1} = \exp\left(-i\tfrac{Lx/(2g)-iQ_2x}{x^2-1} - iH_1 - i\bar{H}_3 + i\bar{H}_4\right)\\
x_1 &= e^{-ip_1} = \exp\left(-i\tfrac{Lx/(2g)+iQ_1}{x^2-1} - iH_1 + iH_2 + i\bar{H}_2 - i\bar{H}_3\right)\\
x_2 &= e^{-ip_2} = \exp\left(-i\tfrac{Lx/(2g)+iQ_1}{x^2-1} - iH_2 + iH_3 + i\bar{H}_1 - i\bar{H}_2\right)\\
y_{\hat{2}} &= e^{-i\hat{p}_2} = \exp\left(-i\tfrac{Lx/(2g)-iQ_2x}{x^2-1} + iH_3 - iH_4 + i\bar{H}_1\right)\\
y_{\hat{3}} &= e^{-i\hat{p}_3} = \exp\left(+i\tfrac{Lx/(2g)-iQ_2x}{x^2-1} - iH_5 + iH_4 - i\bar{H}_7\right)\\
x_3 &= e^{-ip_3} = \exp\left(+i\tfrac{Lx/(2g)+iQ_1}{x^2-1} + iH_6 - iH_5 - i\bar{H}_7 + i\bar{H}_6\right)\\
x_4 &= e^{-ip_4} = \exp\left(+i\tfrac{Lx/(2g)+iQ_1}{x^2-1} + iH_7 - iH_6 - i\bar{H}_6 + i\bar{H}_5\right)\\
y_{\hat{4}} &= e^{-i\hat{p}_4} = \exp\left(+i\tfrac{Lx/(2g)+iQ_2x}{x^2-1} + iH_7 + i\bar{H}_5 - i\bar{H}_4\right) .
\end{aligned}$$

resolvants: $H_a = \sum_{j=1}^{K_a} \frac{x^2}{x^2-1}\frac{1}{x-x_{a,j}}$, $\bar{H}_a(x) = H_a(1/x)$.

- Bohr-Sommerfeld quantization around a finite gap solution from quasimomenta - eigenvalues of the monodromy matrix

$$E-E_{cl} = \oint \frac{dz}{2\pi i}\frac{z}{\sqrt{1-z^2}}\,\partial_z \log\left[\prod_{i=1,2;j=3,4}\frac{(1-y_i/x_j)(1-x_i/y_j)}{(1-x_i/x_j)(1-y_i/y_j)}\right] = \int \frac{du}{2\pi i}\,\partial_u\epsilon_a(u)\log\left(1+Y_{a,0}(u)\right)$$

Gromov, Vieira

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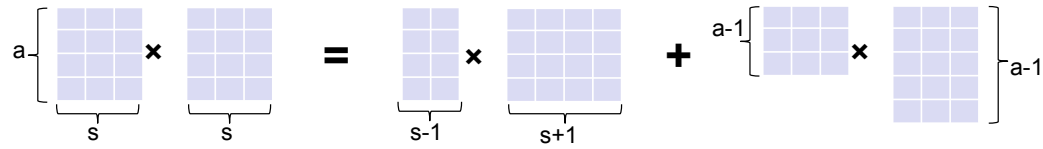
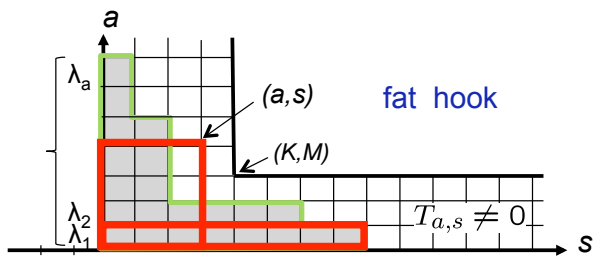
(Super-)group theoretical origins

- A curious property of gl(N) representations with rectangular Young tableaux:



- For characters – simplified Hirota eq.: $T_{a,s}^2 = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$

- Boundary conditions for Hirota eq.: gl(K|M) representations in "fat hook":



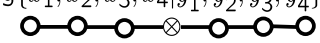
- Solution: Jacobi-Trudi formula for GL(K|M) characters

$$T_{\{\lambda\}}[g] = \det_{1\le i,j\le a} T_{1,\lambda_i - i + j}[g]\,, \qquad g \in GL(K|M).$$

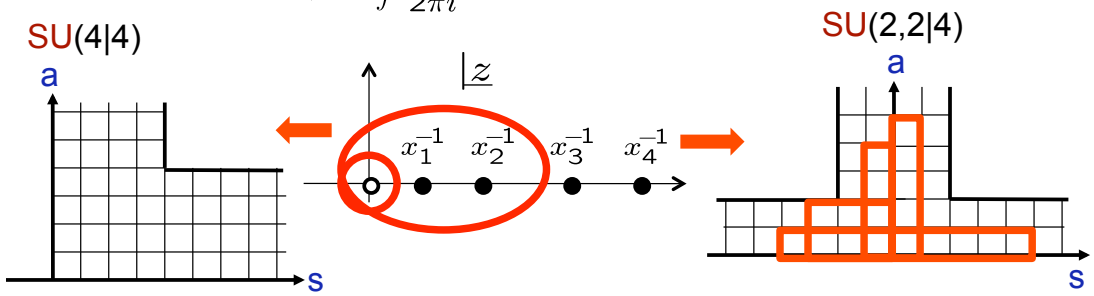
Super-characters: Fat Hook of U(4|4) and T-hook of SU(2,2|4)

- Generating function for symmetric representations:

$$w(z;g) = \mathrm{sdet}(1 - z\,g)^{-1} \qquad g = \mathrm{diag}\{x_1, x_2, x_3, x_4 | y_{\hat1}, y_{\hat2}, y_{\hat3}, y_{\hat4}\} \in SU(2,2|4)$$



$$T_{1,s} = \oint \frac{dz}{2\pi i}\, z^{-s-1}\, w(z;g)$$



∞ - dim. unitary highest weight representations of u(2,2|4) !

Kwon
Cheng,Lam,Zhang
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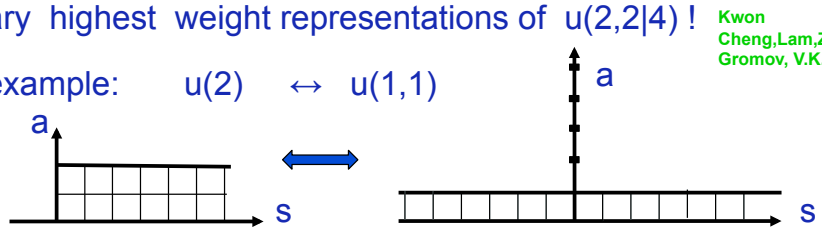
- Amusing example: u(2) ↔ u(1,1)



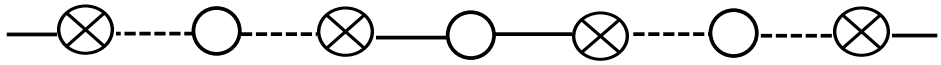
Solving full quantum Hirota in U(2,2|4) T-hook

- Replace gen. function: $w(z;g) = \frac{(1-zy_1)(1-zy_2)}{(1-zx_1)(1-zx_2)}\frac{(1-zy_3)(1-zy_4)}{(1-zx_3)(1-zx_4)} = \sum_s T_{1,s}\, z^s$

  by a generating functional $\quad z^{1/2} \Rightarrow D = e^{-\frac{i}{2}\partial_u}$

Tsuboi
Hegedus
Gromov, V.K., Tsuboi

$$\{y_1|x_1,x_2|y_2,y_3|x_3,x_4|y_4\} \quad \Rightarrow \quad \{Y_1(u)|X_1(u),X_2(u)|Y_2(u),Y_3(u)|X_3(u),X_4(u)|Y_4(u)\}$$



$$W = \left[(1-DY_1D)\frac{1}{(1-DX_1D)}\frac{1}{(1-DX_2D)}(1-DY_2D)\right]_+ \times \left[(1-DY_3D)\frac{1}{(1-DX_3D)}\frac{1}{(1-DX_4D)}(1-DY_4D)\right]_-$$
$$= \sum_{s=-\infty}^{\infty} D^s T_{1,s} D^s$$

$[\cdots]_\pm$ - expansion in $\quad D^{\pm 1} = e^{\mp\frac{i}{2}\partial_u}$

- Parametrization in Baxter's Q-functions:

$$Y_1 = H_R F_0^+ \frac{Q_1^-}{Q_1^+}\ ,\quad X_1 = H_R \frac{Q_1^- Q_2^{++}}{Q_1^+ Q_2}\ ,\quad X_2 = H_R \frac{Q_2^{--} Q_3^+}{Q_2 Q_3^-}\ ,\quad Y_2 = H_R \frac{Q_3^+}{Q_3^-} F_4^-\ ,$$
$$Y_4 = H_L \frac{1}{F_0^-}\frac{Q_7^+}{Q_7^-}\ ,\quad X_4 = H_L \frac{Q_7^+ Q_6^{--}}{Q_7^- Q_6}\ ,\quad X_3 = H_L \frac{Q_6^{++} Q_5^-}{Q_6 Q_5^+}\ ,\quad Y_3 = H_L \frac{Q_5^-}{Q_5^+}\frac{1}{F_4^+}\ .$$

Gromov, V.K., Leurent, Tsuboi

- One can construct the Wronskian determinant solution:
  all T-functions (and Y-functions) in terms of 7 Q-functions

$f(x \pm i/2) \equiv f^\pm$

$f(x+i) \equiv f^{++}$, etc

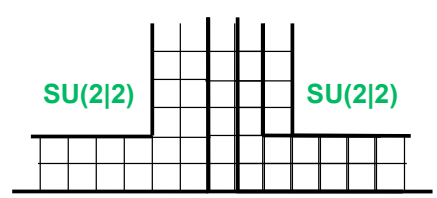
Asymptotic Bethe ansatz solution for Y-system

- Asymptotic S matrix theory also follows from Y-system

$$L \to \infty\,, \qquad \lambda \sim 1\,, \qquad u \sim 1$$

- From TBA: $\quad Y_{a,0} \sim \left(\frac{x^{[-a]}}{x^{[+a]}}\right)^L \to 0$

$$H_L \sim \frac{1}{H_R} \sim \left(\frac{x^{[-]}}{x^{[+]}}\right)^L \to 0$$



SU(2,2|4)

- Parametrization in terms of Bethe roots:

$$F_4 = \prod_j \frac{x - x_{4,j}^+}{x - x_{4,j}^-}\ ,\quad H_R = \left(\frac{x^-}{x^+}\right)^{\frac{L}{2}} \prod_j \frac{x^+ - x_{4,j}^-}{x^- - x_{4,j}^-}\sigma(u, x_{4,j}^\pm)$$

$$F_0 = \bar{F}_4\ ,\quad H_L = \tilde{H}_R$$

$$Q_a = \prod_{j=1}^{K_a}(x(u) - y_{a,j}) \prod_{j=1}^{\bar{K}_a}\left(\frac{1}{x(u)} - y_{\bar{a},j}\right)$$

- Dressing factor follows from the reality of Y-functions.
- Asymptotic Bethe equation: $\quad Y_{1,0}(u_{4,j}) + 1 \simeq 0$

For AdS/CFT, as for any sigma model…

- The origins of the Y-system are entirely algebraic:
  Hirota eq. for characters in a given "hook".

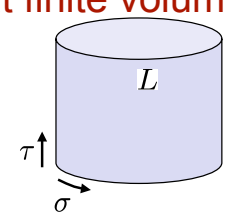
- Add the spectral parameter dependence… and solve Hirota equation!

- Analyticity in spectral parameter u is the most difficult part of the problem.
  Some progress is being made…

Gromov,Tsuboi,V.K.,Leurent
(to appear)

An inspiring example: SU(N) principal chiral field at finite volume

$$\mathcal{S} = \frac{\sqrt{\lambda}}{4\pi}\int d^2x \ \mathrm{Tr}\ \left(g^{-1}\partial_\mu g(x)\right)^2, \qquad g \in SU(N).$$



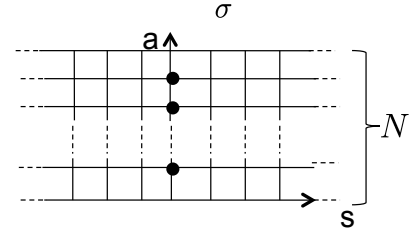
- Y-system ⇨ ***Hirota*** dynamics in a in (a,s) plane.

- General Wronskian solution in a strip:

Krichever,Lipan,
Wiegmann,Zabrodin

$$\mathrm{T}_{a,s}(u) = \begin{vmatrix} \{\bar{\mathrm{Q}}_1^{[-s-2j+1]} & \cdots & \bar{\mathrm{Q}}_N^{[-s-2j+1]} & \}_{j=1,\cdots a} \\ \{\mathrm{Q}_1^{[s-2j-1]} & \cdots & \mathrm{Q}_N^{[s-2j-1]} & \}_{k=a+1,\cdots,N} \end{vmatrix}$$

$$Q(x+\frac{i\,k}{2}) \equiv Q^{[k]}$$



- Finite volume solution: finite system of NLIE:
  parametrization fixing the analytic structure:

Gromov,V.K.,Vieira
V.K.,Leurent

$$\mathrm{Q}_k(u) \ = \ P_k(u) + \int_{-\infty}^{\infty}\frac{dv}{2\pi}\frac{f_k(u)}{u-v}, \qquad \mathrm{Im}(u) < 0\,,$$

polynomials
fixing a state

jumps
by f_k

$$\bar{\mathrm{Q}}_k \longrightarrow \mathrm{Q}_k$$

- N-1 TBA equations fix N-1 spectral densities $f_k(u)$

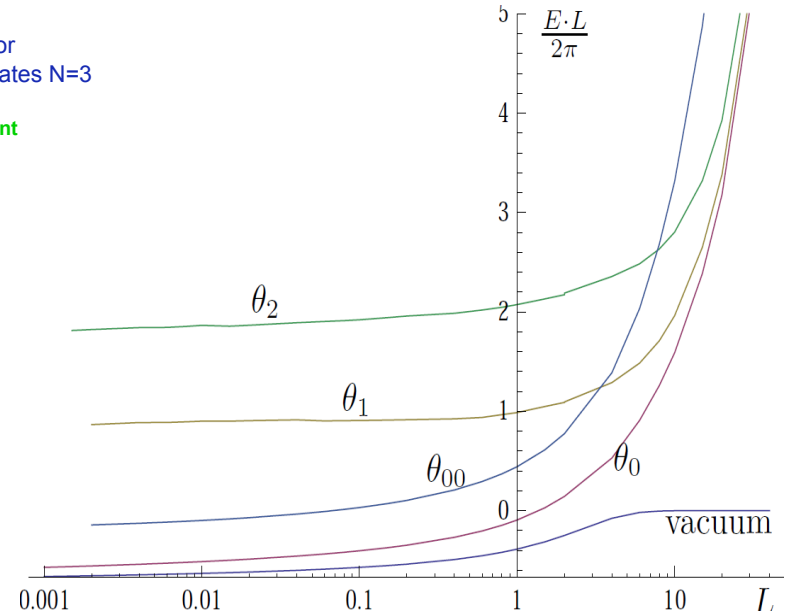
$$\log Y_{a,0} = -Lm_a \sinh(2\pi N u) + K_{a,a'} * \log\left[\frac{(1+Y_{a',1})^2}{(1+Y_{a'+1,0})(1+Y_{a'-1,0})}\right], \qquad \text{where} \quad Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$$

$$a = 1, 2, \ldots, N-1$$

Numerics for $SU(N)_L \times SU(N)_R$ principal chiral field at finite size

Numerics for
low-lying states N=3

V.K.,Leurent



Conclusions

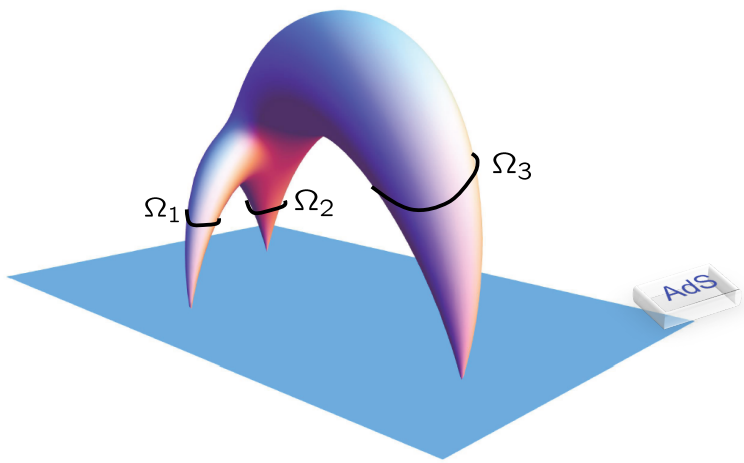
- Non-trivial D=2,3,4,… dimensional solvable quantum field theories!
- Y-system for exact spectrum of a few AdS/CFT dualities has passed many important checks.
- Y-system obeys integrable Hirota dynamics – can be made finite. General method of solving quantum σ-models

Future directions

- Why is N=4 SYM integrable?
- What lessons for QCD?
- 1/N – expansion integrable?
- Amlitudes, correlators … integrable?

3 point function of classical operators

- A generalization of Shapiro-Virasoro amplitudes
- Difficult problem, finite gap method should be considerably advanced
- Classical solution should satisfy the following multiplication rule for monodromy matrices: $\Omega_1(u)\Omega_2(u) = \Omega_3(u)$



…or even 8-point correlators…

END