Associative star-three-product

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Outline



ABJM theory



- Star-three-product
- Three-algebra and star-three-product
- Degrees of freedom
- Connection with usual ABJM
- Abelian M5
 - Deconstruction
- Non-Abelian M5



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How can one

- take the large N limit of hermitian three-algebra?
- deconstruct non-Abelian M5 from ABJM?
- see the $M^{3/2}$ scaling in ABJM?
- see the SO(8) R symmetry in ABJM?
- see the S³ vacuum in mass deformed ABJM?

Hermitian three-algebra

$$[T^a, T^b; T^c] = f^{ab}{}_{cd}T^d$$

describes all ABJM theories.

The three-bracket is complex anti-linear in its third entry, meaning that

$$[T^a, T^b; \lambda T^c] = \lambda^* [T^a, T^b; T^c].$$

The three-bracket takes three elements T^a , T^b and T^c and map these to another element in the algebra.

Hermitian three-algebra is characterized by fundamental identity which says that $[\bullet, T^a; T^b]$ is a derivation, and by trace invariance

$$\left\langle [T^a, T^b; T^c], T^d \right\rangle - \left\langle T^a, [T^d, T^c; T^b] \right\rangle \stackrel{=}{=} 0.$$

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The Bagger-Lambert formulation of ABJM in which the gauge field is expressed as

$$\widetilde{A}^{b}{}_{a} = A^{d}{}_{c}f^{bc}{}_{da}$$

can be used only for $SU(N) \times SU(N)$ gauge group.

To extend the Bagger-Lambert formulation we shall define gauge field as

$$\widetilde{A}^{b}{}_{a} = A^{d}{}_{c}f^{bc}{}_{da} - iA^{i}e_{i}\delta^{b}_{a}.$$

We decompose three-algebra structure constants in trace-less and trace-part

$$f^{bc}{}_{da} = \tilde{f}^{bc}{}_{da} + \lambda \delta^{b}_{a} \delta^{c}_{d}$$

ABJM theory

Here λ is fixed by demanding anti-symmetry

The $U(1)_i$ Chern-Simons levels k_i are constrained by

$$2\pi \sum_{i} \frac{1}{k_i} = \lambda.$$

These constraints result in the following gauge groups and levels:

$$U(N)_k \times U(M)_{-k},$$

 $Sp(N)_k \times U(1)_{-2k},$
 $SU(N)_k \times SU(N)_{-k}.$

These gauge groups correspond to hermitian three-algebras with generators T^a which have unit trace form, complex anti-linear in its second entry,

$$\left\langle T^{a},T^{b}
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angle \ =\ \delta^{a}_{b}$$

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Can we get more gauge groups by including null-generators N^A in the three-algebra? These have trace forms

$$\left\langle N^{A}, T^{a} \right\rangle = 0,$$

 $\left\langle N^{A}, N^{B} \right\rangle = 0.$

The answer is no. It is true that trace invariance puts no restriction on

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but it restricts

$$f^{A\bullet}{}_{\bullet\bullet} = f^{\bullet\bullet}{}_{A\bullet} = 0.$$

The null generators are invisible in the Lagrangian. Also it is still true that f^{ab}_{cd} satisfy the same hermitian fundamental identity as in the absense of null generators.

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Multiplication of functions is associative, but commutative. However we can deform it to a non-commutative star-product

$$\mathcal{F} * \mathcal{G} = \lim_{\sigma \to \sigma'} \exp\left(\frac{i\hbar}{2}\sqrt{g}\epsilon^{lphaeta}\partial_{lpha}\partial_{eta}'
ight) \mathcal{F}(\sigma)\mathcal{G}(\sigma').$$

To linear order in non-commutativity parameter \hbar we have

$$\mathcal{F} * \mathcal{G} = \mathcal{F} \mathcal{G} + \frac{i\hbar}{2} \{\mathcal{F}, \mathcal{G}\} + O(\hbar^2)$$

and we get

$$\begin{aligned} (\mathcal{F}*\mathcal{G})*\mathcal{H} &= \mathcal{F}\mathcal{G}\mathcal{H} + \frac{i\hbar}{2}\{\mathcal{F},\mathcal{G}\}\mathcal{H} + \frac{i\hbar}{2}\{\mathcal{F}\mathcal{G},\mathcal{H}\} + \mathsf{O}(\hbar^2) \\ &= \mathcal{F}\mathcal{G}\mathcal{H} + \frac{i\hbar}{2}(\{\mathcal{F},\mathcal{G}\}\mathcal{H} + \{\mathcal{F},\mathcal{H}\}\mathcal{G} + \{\mathcal{G},\mathcal{H}\}\mathcal{F}) + \mathsf{O}(\hbar^2). \end{aligned}$$

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The result treats $\mathcal{F}, \mathcal{G}, \mathcal{H}$ on the same footing and indeed the star-product is associative

$$(\mathcal{F} * \mathcal{G}) * \mathcal{H} = \mathcal{F} * (\mathcal{G} * \mathcal{H}).$$

Associativity implies that the star-commutator $[\mathcal{F}, \mathcal{G}] = \mathcal{F} * \mathcal{G} - \mathcal{G} * \mathcal{F}$ satisfies the Jacobi identity.

The important thing with the star-product is that it can be used to realize finite-dimensional Lie algebras.

One important example is the fuzzy two-torus.

Star-product

On a two-torus we have two periodic functions

$$\mathcal{U} = \mathbf{e}^{i\sigma^1},$$

 $\mathcal{V} = \mathbf{e}^{i\sigma^2},$

Since

$$[\sigma^1, \sigma^2] = \sigma^1 * \sigma^2 - \sigma^2 * \sigma^1 = i\tilde{\hbar}$$

where

$$\tilde{\hbar} = \hbar \sqrt{g} \epsilon^{12}.$$

we get, using the BCH formula,

$$\mathcal{U} * \mathcal{V} = \mathbf{e}^{i\tilde{h}} \mathcal{V} * \mathcal{U}$$

We define

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which form a Lie algebra

$$[\mathcal{T}^{m_1m_2}, \mathcal{T}^{n_1n_2}] = \left(e^{-i\tilde{\hbar}m_1n_2} - e^{-i\tilde{\hbar}n_1m_2} \right) \mathcal{T}^{m_1+n_1,m_2+n_2}$$

For irrational $\widetilde{\hbar}$ we have infinite dimensional Lie algebras. For

$$\tilde{h} = \frac{2\pi}{N}$$

the structure constants are invariant under $m_{\alpha} \rightarrow m_{\alpha} + N$. Thus we can make a consistent finite truncation of three-algebra generators \mathcal{T}^m and only keep those with ($\alpha = 1, 2$)

$$m_{\alpha} = 0, ..., N - 1.$$

We can then isomorphically map the functions into matrices of size $N \times N$

$$\mathcal{U} \rightarrow U,$$

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$$\mathcal{V} \to V$$

where one choice is

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \ddots & \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\tilde{h}} & 0 \\ & \ddots & \\ 0 & 0 & e^{i(N-1)\tilde{h}} \end{pmatrix}$$

These matrices are unitary,

$$egin{array}{rcl} U^{\dagger} &=& U^{-1},\ V^{\dagger} &=& V^{-1}. \end{array}$$

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and obey

$$egin{array}{rcl} U^N&=&1,\ V^N&=&1,\ UV&=&e^{i{\widetilde{h}}}VU. \end{array}$$

The N^2 different monomials

$$T^{m_1m_2} = U^{m_1}V^{m_2}$$

generate U(N) Lie algebra

$$[T^{m_1m_2}, T^{n_1n_2}] = \left(e^{-i\tilde{\hbar}m_1n_2} - e^{-i\tilde{\hbar}n_1m_2} \right) T^{m+n_2}$$

We can not realize the algebra

$$[\sigma^1, \sigma^2] = \frac{2\pi i}{N}$$

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by finite size matrices.

In the limit $N \to \infty$ we get a commutative torus

$$[\sigma^1, \sigma^2] = 0.$$

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Defining the three-commutator

$$[A, B; C] = AC^{\dagger}B - BC^{\dagger}A$$

we get

$$[[A, B; C], D; E] - [[A, D; E], B; C] - [A, [B, D; E]; C] + [A, B, [C, E; D] = (AC^{\dagger}B)E^{\dagger}D - AC^{\dagger}(BE^{\dagger}D) + \dots$$

Only if we have associativity

$$(AC^{\dagger}B)E^{\dagger}D = A(C^{\dagger}BE^{\dagger})D = AC^{\dagger}(BE^{\dagger}D)$$

we have the fundamental identity.

The first attempt to define a star-three-product would be

$$\mathcal{F} * \mathcal{G} * \mathcal{H} = \mathcal{F} \mathcal{G} \mathcal{H} + rac{\hbar}{2} \{ \mathcal{F}, \mathcal{G}, \mathcal{H} \} + \mathsf{O}(\hbar^2)$$

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but this gives

$$\begin{aligned} (\mathcal{F} * \mathcal{G} * \mathcal{H}) * \mathcal{K} * \mathcal{L} &= \mathcal{FGHKL} \\ &+ \frac{\hbar}{2} (\{\mathcal{F}, \mathcal{G}, \mathcal{H}\} \mathcal{KL} + \{\mathcal{FGH}, \mathcal{K}, \mathcal{L}\}) + O(\hbar^2) \end{aligned}$$

and $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}, \mathcal{L}$ does not appear symmetrically like in the star-product.

When extending the star-product to a star-three-product it is natural to demand this result to be symmetric in $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}, \mathcal{L}$. We extend the ansatz and include additional terms on the form

$$\begin{aligned} \mathcal{F} * \mathcal{G} * \mathcal{H} &= \mathcal{F}\mathcal{G}\mathcal{H} \\ &+ \frac{\hbar}{2} (\{\mathcal{F}, \mathcal{G}, \mathcal{H}\} + \mathbf{a}\mathcal{H}\{\mathcal{F}, \mathcal{G}, \bullet\} + \mathbf{b}\mathcal{F}\{\mathcal{G}, \mathcal{H}, \bullet\} + \mathbf{c}\mathcal{G}\{\mathcal{H}, \mathcal{F}, \bullet\}) \\ &+ O(\hbar^2) \end{aligned}$$

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Associativity determines the three coefficients *a*, *b*, *c* uniquely

$$a = -1, \\ b = -1, \\ c = 1.$$

The odd sign of the last coefficient *c* seems to reflect the fact that the natural product to consider is always $\mathcal{F} * \overline{\mathcal{G}} * \mathcal{H}$ where $\overline{\mathcal{G}}$ denotes complex conjugate.

For matrices this corresponds to $FG^{\dagger}H$. Indeed for $M \times N$ matrices it makes no sense to consider *FGH*.

With a star-three-product we can generalize to a fuzzy three-torus. Again we get a finite-dimensional Lie-three-algebra due to periodic structure constants, invariant under $m_{\alpha} \rightarrow m_{\alpha} + N$ if we choose

$$\hbar = \frac{4\pi}{N}$$

which means that we can consider a consistent finite truncation

$$m_{\alpha} = 0, ..., N - 1.$$

The three-algebra reads,

$$[\mathcal{T}^{m},\mathcal{T}^{n};\mathcal{T}^{p}] = \mathcal{T}^{m+n-p}\left(e^{\frac{i\hbar}{2}\{m,n,p\}}e^{\frac{\hbar}{2}(\{n,p,\bullet\}-\{m,p,\bullet\}+\{m,n,\bullet\})}\right)$$
$$-e^{-\frac{i\hbar}{2}\{m,n,p\}}e^{-\frac{\hbar}{2}(\{n,p,\bullet\}-\{m,p,\bullet\}+\{m,n,\bullet\})}\right)$$

Structure constants are differential operators. The matter fields belong to gauge orbits which are defined as gauge variations of the

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representative $X_m \mathcal{T}^m$. Along this gauge orbit, matter fields may involve a differential operator part, but in each gauge orbit we can, by definition, always find one representative (on the form $X_m \mathcal{T}^m$) that does not involve differential operators.

We define the gauge invariant inner product as

$$\langle X, Y \rangle = \int \frac{d^3\sigma}{(2\pi)^3} (X \cdot 1) (Y \cdot 1),$$

killing any derivatives before integration. Alternatively we consider total derivatives. Either way, derivatives do not contribute to the inner product.

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Classical moduli space are field configurations for which the sextic potential = 0,

$$\mathcal{M} = \{\mathcal{T}^{\vec{m}} | [\mathcal{T}^{\vec{m}}, \mathcal{T}^{\vec{n}}; \mathcal{T}^{\vec{p}}] = 0 \text{ for any } \vec{m}, \vec{n}, \vec{p} \in M\}$$

This in turn implies

$$(\vec{m} \times \vec{n}) \cdot \vec{p} = 0.$$

This means that all vectors \vec{m} must lie in a two-dimensional plane in \mathbb{R}^3 . Taking this plane to be spanned by two coordinate axes we get

$$\dim(\mathcal{M}) = N^2.$$

The number of three-algebra generators is $N^3 = \dim(\mathcal{M})^{2/3}$.

dim(\mathcal{M}) is independent of how we choose the plane. For example take a periodic lattice of points (m_1, m_2) where $m_\alpha = 0, 1, 2, 3 = N - 1$.

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Thus N = 4. Let us take a line starting at the point (0,0) and going through the point (2,3). If we count modulo 4 we now find along this line the following set of points (0,0), (2,3), (0,2), (2,1), (0,0). And from here it repeats itself. The line then goes through four different points. That coincides with the number *N* in this example.

In any three-algebra of the type above we can always find a sub-three-algebra by taking $m_3 = 1$. That is, we consider generators

$$T^{m_1m_21}$$

This gives a sub-three-algebra because the three-bracket involves products like $T^m * T_p * T^n \propto T^{m-p+n}$ and the simple fact that

$$1 - 1 + 1 = 1.$$

The fuzzy three-torus reduces to a fuzzy two-torus and we obtain ABJM theory with gauge group $U(N) \times U(N)$.

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Abelian M5

An M5 in a background C field is described by the Lagrangian density

$$\mathcal{L} = rac{1}{8\pi} \left(-rac{1}{2} H \wedge *H + H \wedge C
ight)$$

plus contributions from the other fields. In this convention we have

$$\int d{m B}~\in~2\pi{\mathbb Z}$$

Once that convention is fixed, the coupling constant $\frac{1}{8\pi}$ is also fixed, by selfduality. Here

$$H = dB + C$$

This construction is made such that only the selfdual part couples to C,

$$\mathcal{L} = rac{1}{8\pi} \left(-rac{1}{2} |dB|^2 - dB \wedge (C + *C) - rac{1}{2} |C|^2
ight)$$

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The action is given by

 $\int d^3x d^3\sigma \sqrt{g}\mathcal{L}$

Since the theory is supersymmetric, we can not add any further constant to the Lagrangian.

- How can we obtain the \sqrt{g} in the measure by deconstruction from ABJM if we have no metric on the auxiliary three-manifold on which we define the star-three-product?
- How can we obtain the coupling constant $\frac{1}{8\pi}$ from ABJM?
- How can we get a background C-field from ABJM?

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Start from ABJM and assume that three real ($X^{\alpha=1,2,3}$) of the eight real ($X^{l=1..8}$) transverse scalar fields are compactified on a three-torus

 $X^{lpha} \sim X^{lpha} + 2\pi R^{lpha}$

If deconstruction around that three-torus gives rise to an M5 brane in a constant background *C*-field, we must among other things, also get the right constant term $-\frac{1}{16\pi}|C|^2$. This is SUSY theory and an arbitrary constant shift of the Lagrangian is not allowed.

The d^3x part is the space-time of the M2. The $d^3\sigma$ comes from the auxiliary (fuzzy) three-torus. Where does \sqrt{g} come from? The inner product is

$$\langle \mathcal{T}^m, \mathcal{T}^n \rangle = \int \frac{d^3\sigma}{(2\pi)^3} \mathcal{T}^m \mathcal{T}_n$$

Deconstruction

unit normalized for functions $\mathcal{T}^m = e^{im_\alpha \sigma^\alpha}$.

Let $Y^{\alpha}R^{\alpha} = X^{\alpha} - 2\pi\sigma^{\alpha}R^{\alpha}$ (no sum) denote fluctuations of scalar fields in transverse space. From the ansatz

$$\mathsf{Y}^lpha ~\propto~ oldsymbol{g} \epsilon^{lphaeta\gamma} oldsymbol{\mathcal{B}}_{eta\gamma}.$$

we get

$$\int d^3\sigma \partial_lpha \, {\sf Y}^lpha ~\propto~ \int d{\cal B}$$

Both sides are 2π quantized. Hence no metric is involved in the proportionality constant here.

The metric arises from identifying the Chern-Simons level by reducing the fuzzy three-torus to a fuzzy two-torus and relate the

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star-three-product with matrices of size $N \times N$. Here *N* corresponds to $U(N) \times U(N)$ gauge group. We then get (for k = 1)

$$\frac{N^2}{4\pi^2\sqrt{g}}$$

as an overall factor multiplying the otherwise canonically normalized BLG action defined with a unit normalized Nambu bracket schematically as

$$\int d^3\sigma \left(-DX'DX'-\{X',X^J,X^K\}\{X',X^J,X^K\}\right).$$

Expanding this action in fluctuations as

$$X' = T' + Y^{\alpha} \partial_{\alpha} T' + ..$$

we get the desired measure factor $d^3\sigma\sqrt{g}$ for the M5 brane. For instance

$$\begin{array}{l} \displaystyle \frac{1}{\sqrt{g}} \partial_{\alpha} T^{I} \partial_{\beta} T^{I} d\mathsf{Y}^{\alpha} d\mathsf{Y}^{\beta} \\ \displaystyle = \ \displaystyle \frac{1}{\sqrt{g}} g_{\alpha\beta} g \epsilon^{\alpha\gamma\delta} g \epsilon^{\beta\epsilon\omega} d\mathsf{B}_{\gamma\delta} d\mathsf{B}_{\epsilon\omega} \\ \displaystyle = \ \displaystyle \sqrt{g} g^{\gamma\epsilon} g^{\delta\omega} d\mathsf{B}_{\gamma\delta} d\mathsf{B}_{\epsilon\omega} \end{array} \end{array}$$

and other terms go the same way. We get the right factor of \sqrt{g} .

The precise numerical value of proportionality constant is fixed by requiring we reproduce the selfdual coupling.

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Doing that we can identify the constant background C-field to be

$$C = \frac{N}{2\pi^2\sqrt{g}} d\sigma^1 \wedge d\sigma^2 \wedge d\sigma^3.$$

The same factor of \sqrt{g}/N occurs in the non-commutativity parameter

$$\hbar = \frac{2\pi}{N}\sqrt{g}.$$

By taking ordinary (non-commutative limit) we thus get infinite C-field, and vice versa.

One must be able to derive the value $\frac{1}{8\pi}$ of the M5 coupling constant from deconstruction. The result I found is that this value of the coupling corresponds to a modified Dirac charge quantization condition

$$\int d{m B}~\in~2\pi{m N}{\mathbb Z}.$$

If I knew that one must have this quantization condition, then I could have said that I had derived the value $\frac{1}{8\pi}$ for the M5 brane coupling.

The number of M2 branes is N^2 , but reducing this to D2 branes by taking $k \to \infty$ we find N D2 and not N^2 . This is due to orbifolding which amounts to reducing $\mathcal{T}^{m_1m_2m_3}$ to $\mathcal{T}^{1m_2m_3}$ by adapting an idea by Nakwoo Kim using Bloch waves on an orbifold. The magnetic charge of the bunch of M2 should be equal to the magnetic charge of the bunch of D2 upon dimensional reduction, hence the magnetic charge should be $2\pi N$ and never $2\pi N^2$.

We take three-algebra generators as

$$\mathcal{T}^{mm'}(\sigma,\sigma') = \mathcal{T}^{m}(\sigma)\mathcal{T}^{m'}(\sigma')$$

and we define the star-three-product on this tensor product as

$$(\mathcal{T}^{mm'}*\mathcal{T}_{pp'}*\mathcal{T}^{nn'})_{\mathcal{AB}} = (\mathcal{T}^m*\mathcal{T}_p*\mathcal{T}^n)_{\mathcal{A}}(\mathcal{T}^{m'}*\mathcal{T}_{p'}*\mathcal{T}^{n'})_{\mathcal{B}}.$$

We can have different non-commutativity parameters for ${\cal A}$ and ${\cal B}$ products,

and N_A will count the degree of fuzziness while N_B will give the non-Abelian structure on M5. Number of three-algebra generators in \mathcal{B} is N_B^3 and this counts the degrees of freedom on M5. We shall take $N_A = \infty$ unless we are interested in non-commutative M5. Still, on a three-torus we get a *C*-field.

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 S^3/\mathbb{Z}_k

If $Z^{\mathsf{A}} \in \mathbb{C}^4$, orbifolding $\mathbb{C}^4/\mathbb{Z}_k$ acts like

$$Z^A \rightarrow e^{\frac{2\pi i}{k}}Z^A.$$

Since the fiber direction on S^3 is the phase, $G^a = e^{i\psi}\widetilde{G}^a$, the orbifolding amounts to orbifolding the fiber as S^1/\mathbb{Z}_k . (Here $Z^A = (G^a, G^{\dot{a}})$ and $4 \rightarrow 2 + 2$ of $SU(4) \rightarrow SU(2) \times SU(2)$.)

Orbifolding enable us to interpolate between D4 and M5.

Once one orbifolds transverse space the total antisymmetry of the three-bracket in the ABJM Lagrangian is broken, and we can no longer use the BLG form, but have to use the ABJM form. Taking $k \to \infty$ we get D4 on S^2 base of S^3 as viewed as a Hopf fibration. Taking k = 1 we get M5 on S^3 .

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S^3/\mathbb{Z}_k

The GRVV equation

$$G^a = [G^a, G^b; G^b]$$

only gives fuzzy S^2 if we use matrices. We find S^3 if we use embedding functions $\mathcal{G}^1 = X^1 + iX^2$ and $\mathcal{G}^2 = X^3 + iX^4$. In a special case we have the Nambu bracket and a classical S^3 ,

$$\begin{array}{rcl} \mathcal{G}^{a} & = & \{\mathcal{G}^{a}, \mathcal{G}^{b}, \mathcal{G}_{b}\} \\ & \Leftrightarrow \\ e^{ijkl} X^{l} & = & \{X^{i}, X^{j}, X^{k}\}. \end{array}$$

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