String Dualities and Ultraviolet Behaviour of Supergravity

Michael B. Green
DAMTP, Cambridge

"Gauge Theories And The Structure Of Spacetime "
The Orthodox Academy of Crete, Kolymvari,

September 15 2010

I) Non-perturbative aspects of maximally supersymmetric closed string theory scattering amplitudes.

Low energy expansion in flat space in D < 11 dimensions. Duality symmetries.

1) MBG, Jorge Russo, Pierre Vanhove,
"Automorphic properties of low energy string amplitudes in various dimensions" arXiv:1001.2535;

2) MBG, Stephen Miller, Jorge Russo, Pierre Vanhove,
"Eisenstein series for higher-rank groups and string theory amplitudes", arXiv:1004.0163

Also: Gutperle, Kiritsis, Pioline, Obers, Sethi, Basu, ...

II) Relationship to multiloop supergravity quantum field theory.

Ultraviolet divergences in D dimensions.

3) MBG, Jorge Russo, Pierre Vanhove,
"String theory dualities and supergravity divergences", arXiv:1002.3805

III) Perturbative supergravity using pure spinor quantum mechanics.

Explicit evidence for UV divergence of form d^8R^4 in maximal supergravity.

4) Jonas Bjornsson, MBG "5 loops in 24/5 dimensions", arXiv:1004.2692

I) Dualities of type II theory on d=(10-D)-torus $\mathcal{T}^d$

Duality invariance implies relations between perturbative and nonperturbative terms in the S-matrix.

Non-trivial dependence on the moduli (scalar fields),

- in contrast to classical supergravity

Moduli space : space of scalar fields

Scalar fields in maximally supersymmetric supergravity

discrete identification- breaks symmetry to discrete subgroup

maximal compact subgroup

$$G(\mathbb{Z})\backslash \overbrace{G(\mathbb{R})/K}$$

Duality Group

U-duality groups for type II in D dimensions

on $\mathcal{T}^d$ $\quad D = 10 - d$

Dimension D	$G_d(\mathbb{Z})$
$10A$	1
$10B$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{Z})$
8	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{Z})$
6	$SO(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}(\mathbb{Z})$

decompactification

. . . . . .

Four (super)-Graviton Scattering in type II

(in D dimensions)

$$A_D(s,t,u) = A_D^{analytic}(s,t,u) + A_D^{nonan}(s,t,u)$$

local term in eff. action

contains massless thresholds
- nonlocal terms in eff. action
(IR divergent for $D \leq 4$).

$$A_D^{analytic}(s,t,u) = \mathcal{R}^4\, T_D(s,t,u)$$

$$s = -(k_1+k_2)^2, \quad t = -(k_1+k_4)^2, \qquad u = -(k_1+k_3)^2$$

$\mathcal{R}$ Linearised supercurvature – describes 256 physical states in supermultiplet.

Momentum expansion:

$$T_D(s,t,u) = \sum_{p,q} \mathcal{E}^{(D)}_{(p,q)}\, \sigma_2^p\, \sigma_3^q$$

Power series in

$$\sigma_2 = s^2 + t^2 + u^2$$
$$\sigma_3 = s^3 + t^3 + u^3$$

Coefficients are duality invariant functions of moduli

Duality - invariant effective action

(Einstein frame) Einstein-Hilbert Higher-derivative interactions

$$S = \frac{1}{\ell_D^{D-2}} \int d^D x \sqrt{-G^{(D)}}\, R + S^{local} + \ldots$$

$\sigma_2\, \mathcal{R}^4$ $\sigma_3\, \mathcal{R}^4$

$$S_D^{local} = \ell_D^{8-D} \int d^D x \sqrt{-G^{(D)}} \left(\mathcal{E}^{(D)}_{(0,0)}\, \mathcal{R}^4 + \ell_D^4\, \mathcal{E}^{(D)}_{(1,0)}\, \partial^4 \mathcal{R}^4 + \ell_D^6\, \mathcal{E}^{(D)}_{(0,1)}\, \partial^6 \mathcal{R}^4 + \ldots \right)$$

1/2 BPS 1/4 BPS 1/8 BPS

Planck length

Is this the complete list of "protected" terms??

Coefficients $\mathcal{E}^{(D)}_{(p,q)}$ satisfy Laplace equations in various dimensions (motivated by **SUSY**): MBG, Russo, Vanhove 2010

$\Delta^{(D)} \equiv \Delta_{E_{11-D}/K_D}$ Invariant laplacian on coset space

R^4

$$\left(\Delta^{(D)} - \frac{3(11-D)(D-8)}{D-2}\right)\mathcal{E}^{(D)}_{(0,0)} = 6\pi\,\delta_{D-8,0}$$

Singularity in D=8

"critical" dimensions

$\partial^4 R^4$

$$\left(\Delta^{(D)} - \frac{5(12-D)(D-7)}{D-2}\right)\mathcal{E}^{(D)}_{(1,0)} = 40\zeta(2)\,\delta_{D-7,0}$$

Singularity in D=7

$\partial^6 R^4$

$$\left(\Delta^{(D)} - \frac{6(14-D)(D-6)}{D-2}\right)\mathcal{E}^{(D)}_{(0,1)} = -\left(\mathcal{E}^{(D)}_{(0,0)}\right)^2 + 120\zeta(3)\,\delta_{D-6,0}$$

Singularity in D=6

R^4 source

e.g. D=10 IIB duality group $SL(2,\mathbb{Z})$

modulus $\tau = \tau_1 + i\tau_2$

coupling $\tau_2 \equiv e^{-\phi} = g_B^{-1}$

$$\left(\Delta^{(10)} - 3/4\right) \mathcal{E}^{(10)}_{(0,0)} = 0 \qquad \left(\Delta^{(D)} - 15/4\right) \mathcal{E}^{(10)}_{(1,0)} = 0$$

$s = 3/2$ $\qquad$ $s = 5/2$

Laplace eigenvalue equations $\quad \Delta_\tau E_s = s(s-1) E_s$

Unique solutions $\quad E_s^{SL(2)} \equiv E_s = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}}$

nonholomorphic Eisenstein series

$$\sim 2\zeta(2s)\tau_2^s + (\dots)\zeta(2s-1)\tau_2^{1-s} + \sum_{k\neq 0} \mu(k,s) \left(e^{2\pi i k\tau} + c.c.\right)\left(1 + O(\tau_2^{-1})\right)$$

TREE-level term $(\tau_2 = g_s^{-1})$

GENUS-$(s - \frac{1}{2})$ term

D-INSTANTON terms
Infinite no. of pert. Corrections-
non-zero Fourier modes.
Interesting measure

NO HIGHER LOOP TERMS

$\mathcal{R}^4$ $\qquad$ $\partial^4\mathcal{R}^4$

tree-level + one-loop $\qquad$ tree-level + two-loop

Higher order: $\mathcal{E}^{(10)}_{(0,1)}\,\sigma_3\,\mathcal{R}^4 \sim \mathcal{E}^{(10)}_{(0,1)}\,\partial^6\,\mathcal{R}^4$

Satisfies inhomogeneous Laplace eigenvalue equation

R^4 source

$$(\Delta_\tau - 12)\,\mathcal{E}^{(10)}_{(0,1)} = -E_{\frac{3}{2}}\,E_{\frac{3}{2}} = -\left(\mathcal{E}^{(10)}_{(0,0)}\right)^2$$

- Motivated by four-graviton scattering amplitude, and consistent with supersymmetry expectations (but not yet derived from SUSY).

- Contains tree, 1-loop, 2-loop, 3-loop perturbative contributions

General D involves :

Eisenstein series for higher-rank groups

General Eisenstein series depends on $r = \mathrm{rank}\ G$ parameters $s_i \quad i = 1, \ldots, r$ (Selberg, Langlands)

Ignoring (significant) details, denote this E_s^G

These series satisfy Laplace equations - quadratic Casimir

$$(\Delta_{G/K} - \lambda)\, E_s^G = 0$$

eigenvalue depends on $G,\ s$

SOLUTIONS to Laplace equations and b.c.'s:

MBG, Miller, Russo, Vanhove 2010
(Pioline 2010)

e.g. $D = 3, \ (E_8)$

$$R^4 \qquad \mathcal{E}^{(3)}_{(0,0)} = E^{E_8}_{\frac{3}{2}}$$

$$\partial^4 R^4 \qquad \mathcal{E}^{(3)}_{(1,0)} = E^{E_8}_{\frac{5}{2}}$$

Particular Eisenstein series for E_8

More complicated for

$$\mathcal{E}^{(3)}_{(0,1)} \, \partial^6 \mathcal{R}^4$$

Laplace eigenvalue equations and boundary conditions.

Precise perturbative expansions in three limits:

(i) Large radius limit - decompactify from D to D+1 dimensions;

(ii) String perturbation theory limit – genus 0,1,2,3;

(iii) Large M-theory torus limit –
11-dim. SUGRA Feynman diagrams 1,2 loops on (11-D)-torus

Detailed agreement of very many coefficients !!

Note:

For all $3 \leq D \leq 10$ higher-rank duality groups,

R^4 not renormalised beyond one loop.

$\partial^4 R^4$ not renormalised beyond two loops.

$\partial^6 R^4$ not renormalised beyond three loops.

What about $\partial^8 R^4$???

Does it get contributions beyond four loops ???

(see later)

II) Determining Supergravity UV divergences

MBG, Russo, Vanhove 2010

Maximal SUGRA has logarithmic UV divergences in the following 'Critical' dimensions: (poles in ϵ)

protected F-terms

- One loop in D=8 R^4
- Two loops in D=7 $\partial^4 R^4$
- Three loops in D=6 $\partial^6 R^4$
- Four loops in D=11/2 $\partial^8 R^4$

$$D = 4 + 6/L$$

Five loops in D=??? $\partial^{??} R^4$

These critical logarithms are encoded precisely in the duality invariant coefficients.

(i) D=8 R^4 Duality group $SL(3) \times SL(2)$ (moduli T, U, y_8)

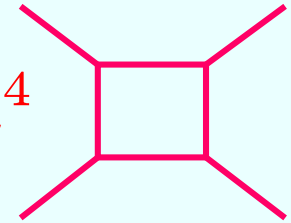
$y_8 =$ String coupling in D=8

$$\mathcal{E}^{(8)}_{(0,0)} \sim \frac{2\zeta(3)}{y_8} + 2\hat{E}_1^{SL(2)}(T) + 2\hat{E}_1^{SL(2)}(U) + \frac{2\pi^2}{3}\log y_8 + n.p.$$

tree 1-loop 1-loop logarithm D-instantons

- String perturbation theory (in **Einstein** frame)

Recall maximal SUGRA R^4



1-loop logarithm

$\mathcal{R}^4 \log s$

$\log y_8$ must come from $\log(s\, l_s^2)$ in the string frame

$$\log(s\, l_s^2) = \log(s\, l_D^2) - \frac{1}{D-2} \log y_D$$

Weyl dilaton factor $l_s^{D-2} = l_D^{D-2}\, y_D^{-1}$

Coefficient is determined by unitarity and must be the same as the coefficient of $c\, \log(s/\Lambda^2)$ in supergravity.

Ultraviolet cutoff in supergravity

The coefficient of $\log y_8$ in $\mathcal{E}^{(8)}_{(0,0)}$ determines the UV logarithm in one-loop maximal supergravity in D=8 R^4

Similarly for the other SUGRA log divergences:

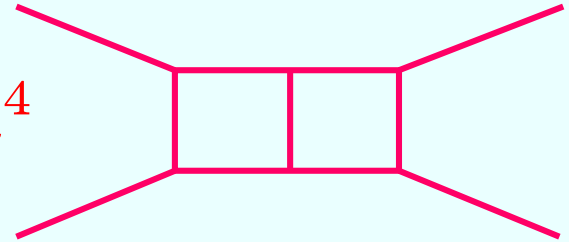
(ii) D=7 $\partial^4 R^4$ Duality group $SL(5)$

$$\mathcal{E}^{(7)}_{(1,0)} \sim \cdots + \frac{8\pi^2}{15} \log y_7$$

2-loop logarithm

Agrees with two-loop supergravity UV divergence in D=7

$S^2\, R^4$



Bern, Dixon, Dunbar, Perelstein, Rozowsky 1998

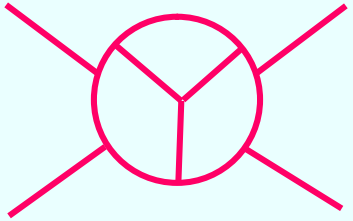
(iii) D=6 $\partial^6 R^4$ Duality group $SO(5,5)$

$$\mathcal{E}^{(6)}_{(0,1)} \sim \cdots + 15\zeta(3) \log y_6$$

3-loop logarithm

Agrees with three-loop supergravity UV divergence in D=8 *

$S^3\, R^4$



Bern, Carrasco, Dixon, Johansson, Kosower, Roiban 2007

* Factor of 6 missing??

BUT

Supergravity limit of string theory

Ooguri, MBG, Schwarz 2006
MBG, Russo, Vanhove 2010

Whether or not there are ultraviolet divergences, supergravity does not decouple from string theory.

The supergravity limit:

Fixed $\ell_D^{D-2} = \ell_s^{D-2} y_D$ $\ell_s \to 0$ (decouples string excitations)

i.e. $y_D \to \infty$

Infinite towers of charged BPS black hole states become massless and/or instantons have zero action.

Wrapped p-branes; KK charges; KK monopoles.

Comments on $\partial^8 \mathcal{R}^4$:

- Four-loop supergravity UV divergence in $D = \frac{11}{2}$

Bern, Carrasco, Dixon, Johansson, Roiban 2009

Is there a FIVE-LOOP contribution ??

- Indications of genus $h = 5$ contribution to $\partial^8 \mathcal{R}^4$

MBG, Russo, Vanhove 2010

Duality argument — MBG, Russo, Vanhove 2008

Pure spinor string — Berkovits, MBG, Russo, Vanhove 2009

*** Pure spinor field theory — Bjornsson, MBG 2010

- Suggests $\partial^8 \mathcal{R}^4$ is a "D-term" contributions from all h

Would lead to **SEVEN**-LOOP UV divergence in N=8 SUGRA

(also suggested by certain superspace arguments) $D = 4$

(NOTE: For $\partial^{2k} \mathcal{R}^4$: If $h \leq k$ for all k, UV divergences absent for $D < 4 + \frac{6}{h}$)

III) Explicit 5-loop contribution to $\partial^8 \mathcal{R}^4$

Jonas Bjornsson, MBG "5 loops in 24/5 dimensions", arXiv:1004.2692

Based on a pure spinor formalism for the superparticle.

World-line formalism with coordinates: manifest supersymmetry

Bosons: World-line scalars $X^m, \lambda^\alpha, \bar{\lambda}_\alpha$

16, 11, 11 no. of zero modes

World-line vectors $P_m, w_\alpha, \bar{w}^\alpha$

16L, 11L, 11L L= loop number

Fermions: World-line scalars θ^α, r_α

16, 11

World-line vectors d_α, s^α

pure spinors $\lambda \gamma^m \lambda = 0$, $\lambda \gamma^m r = 0$ $\lambda, \bar{\lambda}, w, \bar{w}, r, s$ 16L, 11L

For supergravity these are doubled $\hat{\theta}, \hat{\lambda}, \hat{\bar{\lambda}}, \hat{w}, \hat{\bar{w}}, \hat{r}, \hat{s}$

L-loop amplitude :

$$\int \mathcal{D}\Phi\mathcal{D}\hat{\Phi}\,\mathcal{N}\hat{\mathcal{N}}\prod_{i=1}^{3(L-1)}\left(\int_0^{T_i}\frac{d\tau}{T_i}b\int_0^{T_i}\frac{d\tau}{T_i}\hat{b}\right)\int\prod_{r=1}^{4}d\tau_r\,V(k_1,\tau_1)\ldots V(k_4,\tau_4)\,e^{-S}\Bigg)$$

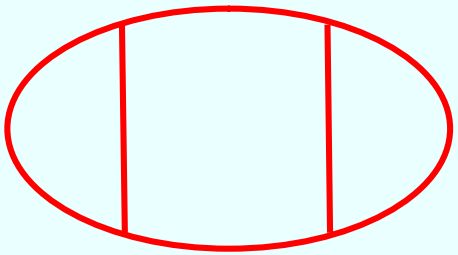
Regulator for large $\lambda\,,\bar{\lambda}$ divergences $\mathcal{N}=e^{-\lambda\bar{\lambda}+\theta r-\lambda\,d\,s+\ldots}$

(3L-3) $b\,,\hat{b}$ insertions in L-loop amplitude

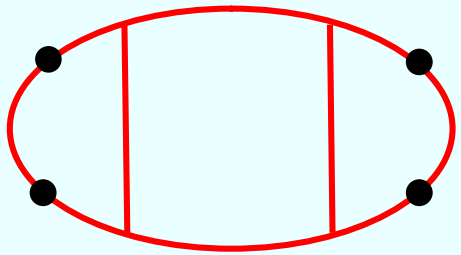
Saturation of fermionic zero modes. Requires detailed consideration of modes coming from regulator $\mathcal{N}\,\hat{\mathcal{N}}$, b insertions and vertex insertions, V .

Constrains pattern of diagrams that contribute to amplitude.

Three-loop amplitude



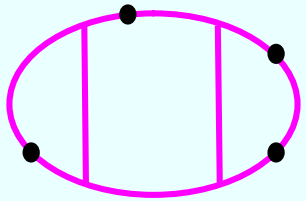
$\partial^8 \mathcal{R}^4 \Lambda^{3D-20}$



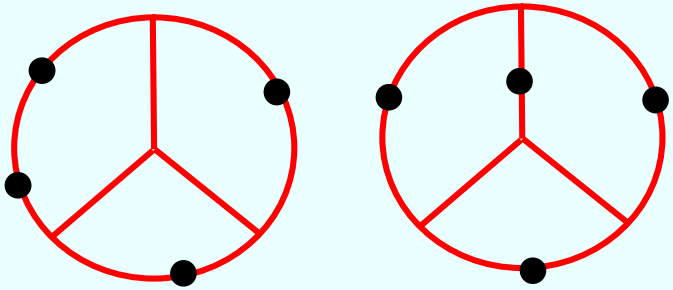
Ladder amplitude

The three-loop skeletons

Not allowed



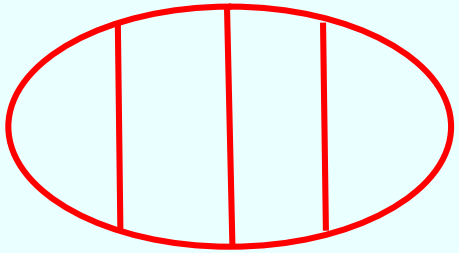
$\partial^6 \mathcal{R}^4 \Lambda^{3(D-6)}$



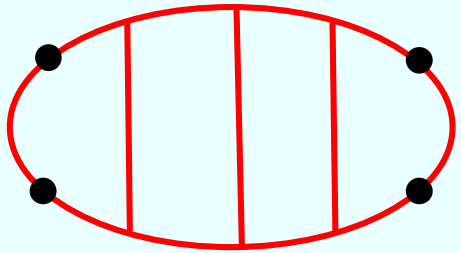
Two leading contributions

Agrees with diagrams in Bern, Carrasco, Dixon, Johansson, Kosower, Roiban 2007

Four-loop amplitude

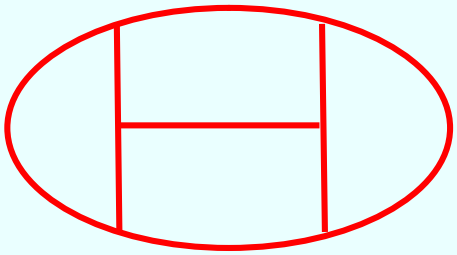


$$\partial^{12}\,\mathcal{R}^4\,\Lambda^{4D-26}$$

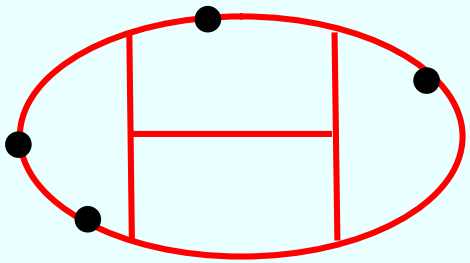


Ladder amplitude

Two of the five
four-loop skeletons



$$\partial^{8}\,\mathcal{R}^4\,\Lambda^{4D-22}$$



Example of a leading amplitude

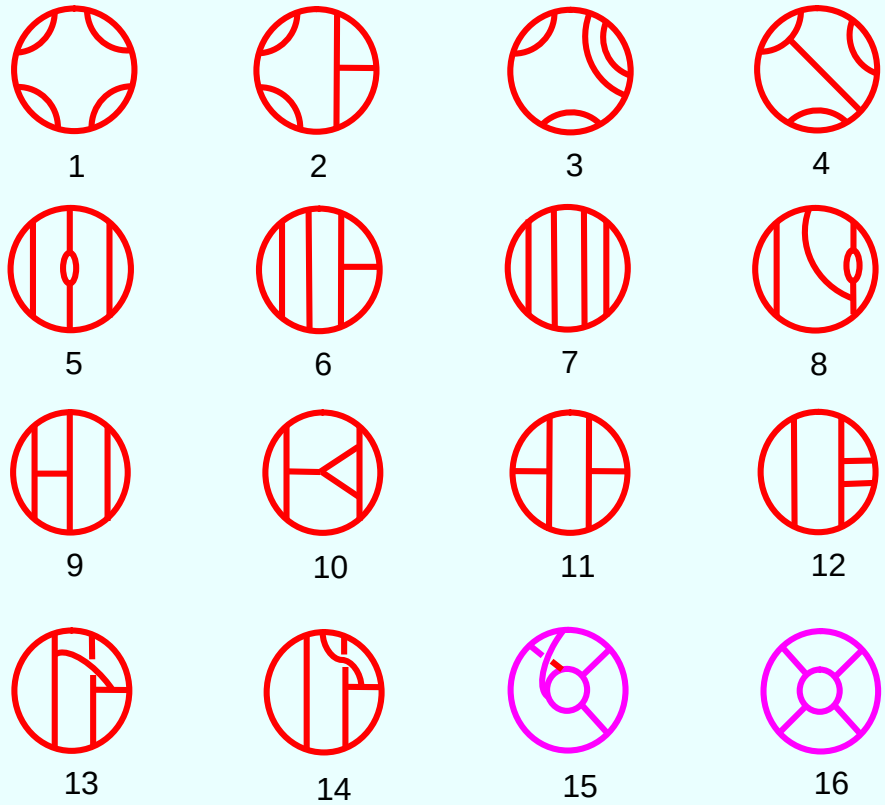
Agrees with diagrams in Bern, Carrasco, Dixon, Johansson, Roiban 2009

FIVE loops

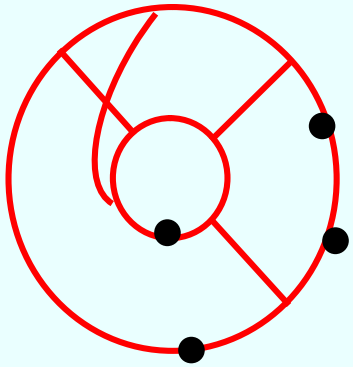
Requires 3L-3 = 12 b insertions. Small- λ , $\bar{\lambda}$ singularity.
Needs new regulator (Berkovits-Nekrasov) changes systematics.

Five-loop skeletons - to which vertices must be inserted

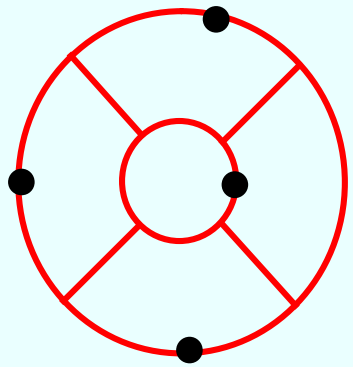
skeletons



Last two diagrams give leading contribution:



$$\partial^8 \mathcal{R}^4 \Lambda^{5D-24}$$



Nonplanar

Planar

Momentum factors in vertices cancel adjacent propagators, leading to degree of UV divergence.

5 loops k^{5D}

12 propagators k^{-24}

Log UV divergence in D=24/5 dimensions proportional to

$$\partial^8 \mathcal{R}^4 \log \Lambda$$

Furthermore, general arguments suggest higher loops also contribute to $\partial^8 \mathcal{R}^4$ so this interaction is not protected.

c.f. If $\partial^8 \mathcal{R}^4$ were protected the low energy five-loop ampltude would have behaved as

$$\partial^{10} \mathcal{R}^4 \Lambda^{5D-26}$$

consistent with $D = 4 + \frac{6}{L}$ and UV finiteness of N=8 supergravity in D=4

Strongly suggests that supersymmetry protects interactions of the form $\partial^{2k} \mathcal{R}^4$ up to k=3. The interaction $\partial^8 \mathcal{R}^4$ is unprotected and likely to have an ultraviolet divergence at seven loops in D=4 dimensions.