Holographic Hydrodynamics :
A radial flow of Response function

Nabamita Banerjee
ITF, Utrecht

Crete Conference, 2010

JHEP 1008:041,2010 (with S Dutta)
arXiv:1008.3852 [hep-th] (with D Astefanesei and S Dutta)

Hydrodynamic system

- *Long wavelength* effective description of strongly coupled field theory.
- It is formulated in the language of constituent equations.
- The simplest case: no global conserved currents

$$\nabla_\mu T^{\mu\nu} = 0$$

- Need to reduce the number of independent elements of $T^{\mu\nu}$.

Hydrodynamic system

- We consider the fluid is in local thermal equilibrium and fluctuations are of small energy.
- At any given time the system is described by the following local quantities

Temperature: $T(x)$ $\hookrightarrow 1$
Velocity vector: $u^{\mu}(x)$ $\hookrightarrow d+1$

- Velocity vectors are normalized : $u^{\mu}u_{\nu} = -1$

■ Total $d+1$ variables.

- In hydrodynamics we express $T^{\mu\nu}$ through $T(x)$ and $u^{\mu}(x)$ through the so-called constitutive equations.

Hydrodynamic system

0^{th} order

$$T^{\mu\nu} = (\epsilon + P)u^{\mu}u^{\nu} + Pg^{\mu\nu}$$

$\epsilon \rightarrow$ energy density and $P \rightarrow$ pressure.

Hydrodynamic system

1^{st} order

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} - \sigma^{\mu\nu}.$$

$$\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left[\eta\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{3}g_{\alpha\beta}\nabla_\lambda u^\lambda\right) + \zeta g_{\alpha\beta}\nabla_\lambda u^\lambda\right].$$

$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \rightarrow$ Projection operator.

$\eta \rightsquigarrow$ shear viscosity coefficient
$\zeta \rightsquigarrow$ bulk viscosity coefficients

For conformal fluid $\zeta = 0$.

Hydrodynamic system

2^{nd} order

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} - \sigma^{\mu\nu} + \Theta^{\mu\nu}.$$

$$\begin{aligned}\Theta^{\mu\nu} = \eta \tau_\Pi & \left[{}^{\langle} D \sigma^{\mu\nu \rangle} + \frac{1}{d-1} \sigma^{\mu\nu} (\nabla \cdot u) \right] \\ & + \kappa \left[R^{\langle \mu\nu \rangle} - (d-2) u_\alpha R^{\alpha \langle \mu\nu \rangle \beta} u_\beta \right] \\ & + \lambda_1 \sigma^{\langle \mu}{}_\lambda \sigma^{\nu \rangle \lambda} + \lambda_2 \sigma^{\langle \mu}{}_\lambda \Omega^{\nu \rangle \lambda} + \lambda_3 \Omega^{\langle \mu}{}_\lambda \Omega^{\nu \rangle \lambda} .\end{aligned}$$

- S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, Nonlinear Fluid Dynamics from Gravity, JHEP 0802, 045 (2008) [arXiv:0712.2456 [hep-th]].
- R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, Relativistic viscous hydrodynamics, conformal invariance, and holography, arXiv:0712.2451 [hep-th].
- M. Natsuume and T. Okamura, Causal hydrodynamics of gauge theory plasmas from AdS/CFT duality, Phys. Rev. D 77, 066014 (2008) [arXiv:0712.2916 [hep-th]].

Kubo formula: Transport coefficients from thermal correlators

- Consider the response of the fluid to small and smooth metric perturbations: $g_{xy} = \eta_{xy} + h_{xy}(t,z)$

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} - \sigma^{\mu\nu} + \Theta^{\mu\nu}$$

$$\Downarrow$$

$$T^{xy} = -Ph_{xy} - \eta \dot{h}_{xy} + \eta\tau_\Pi \ddot{h}_{xy} - \frac{\kappa}{2}[(d-3)\ddot{h}_{xy} + \partial_z^2 h_{xy}] .$$

- In linear response theory

$$\langle T_{xy} \rangle \sim G^R_{xy,xy} h_{xy}$$

$$G_R^{xy,xy}(\omega, k) = P - i\eta\omega + \eta\tau_\Pi\omega^2 - \frac{\kappa}{2}[(d-3)\omega^2 + k^2] .$$

Holographic computation of Green's function

- We start with five dimensional action

$$S_{\mathrm{EM}} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g}\,(R + 12)\,.$$

- The background has a black-brane solution as,

$$\begin{aligned} dS^2 &= -g_{tt}dt^2 + g_{rr}dr^2 + g_{ij}dx^2dx^j, \\ g_{tt} &= r^2(1 - \frac{1}{r^4}), \quad g_{rr} = \frac{1}{g_{tt}} \\ g_{ij} &= r^2\delta_{ij}. \end{aligned}$$

- Solution is asymptotically AdS and boundary topology $\rightarrow$ $R \times R^3$.
- The horizon is at $r \rightarrow 1$ and asymptotic boundary is at $r \rightarrow \infty$.

Holographic computation of Green's function

- We study the graviton's fluctuation in this background,

$$g_{xy} = g_{xy}^{(0)} + h_{xy}(r,x) = g_{xy}^{(0)}[1 + \epsilon\Phi(r,x)].$$

- Plugging it in the action and keeping terms to order ϵ^2,

$$\begin{aligned} S &= \int \frac{d^4k}{(2\pi)^4} dr(\mathcal{A}_1(r,k)\phi'(r,k)\phi'(r,-k) \\ &\quad + \mathcal{A}_0(r,k)\phi(r,k)\phi(r,-k)) \end{aligned}$$

$$\Phi(r,x) = \int \frac{d^4k}{(2\pi)^4} e^{ik.x}\phi(r,k)$$

- The coefficients $\mathcal{A}_1(r,k)$ and $\mathcal{A}_0(r,k)$ are given by

$$\begin{aligned}\mathcal{A}_1(r,k) &= -\frac{\frac{1}{2}g^{rr}\sqrt{-g}}{16\pi G_5},\\ \mathcal{A}_0(r,k) &= -\frac{\frac{1}{2}\sqrt{-g}\,g^{\mu\nu}k_\mu k_\nu}{16\pi G_5}.\end{aligned}$$

Holographic computation of Green's function

$$
\begin{aligned}
S &= \int \frac{d^4k}{(2\pi)^4} dr(\mathcal{A}_1(r,k)\phi'(r,k)\phi'(r,-k) \\
&\quad + \mathcal{A}_0(r,k)\phi(r,k)\phi(r,-k))
\end{aligned}
$$

- Conjugate momentum of the transverse graviton

$$
\begin{aligned}
\Pi(r, k_\mu) &= \frac{\partial S}{\partial \phi'(r,k)} \\
&= 2\mathcal{A}_1(r,k)\phi'(r,k)
\end{aligned}
$$

- The equation of motion

$$
\Pi'(r, k_\mu) - 2\ \mathcal{A}_0(r,k)\phi(r,k) = 0\ .
$$

Holographic computation of Green's function

- Computing the gravitons action on-shell it reduces to the following surface term

$$S = \int \frac{d^4k}{(2\pi)^4} (\mathcal{A}_1(r,k)\phi'(r,k)\phi(r,-k))\Big|_1^\infty .$$

- Following the Minkowskian prescription (Son-Starinets), the boundary retarded Green's function is given as,

$$G_R(k_\mu) = \lim_{r\to\infty} -\frac{2\mathcal{A}_1(r,k)\phi'(r,k)\phi(r,-k)}{\phi_0(k)\phi_0(-k)}$$

- $\phi_0(k_\mu)$ is the value of the graviton fluctuation at boundary.

A response function

- We can rewrite the boundary retarded Green's function as,

$$G_R(k_\mu) \quad = \quad \lim_{r\to\infty} -\frac{\Pi(r,k_\mu)}{\phi(r,k_\mu)}.$$

- Define a response function of the boundary theory

$$\bar{\chi}(k_\mu, r) = \frac{\Pi(r,k_\mu)}{i\omega\phi(r,k_\mu)} \qquad \omega = k_0$$

- This function is defined for all r and k_μ.
- Therefor the boundary Green's function is given by,

$$G_R(k_\mu) = \lim_{r\to\infty} -i\omega\bar{\chi}(k_\mu, r).$$

Radial evolution of the response function

- Differentiate the response function w.r.t. r

$$\partial_r \bar{\chi}(r, k_\mu) = \frac{1}{i\omega}\left[\frac{\Pi'(r,k)}{\phi(r,k)} - \frac{\Pi(r,k)\phi'(r,k)}{\phi(r,k)^2}\right]$$

- Using the definition of $\Pi(r, k)$ and field equation of motion

$\Pi(r, k_\mu) = 2\mathcal{A}_1(r,k)\phi'(r,k) \quad , \quad \Pi'(r, k_\mu) - 2\,\mathcal{A}_0(r,k)\phi(r,k) = 0 \; .$

$$\partial_r \bar{\chi}(k_\mu, r) = i\omega\sqrt{-\frac{g_{rr}}{g_{tt}}}\left[\frac{\bar{\chi}(k_\mu, r)^2}{\Sigma(r)} - \frac{\Upsilon(r)}{\omega^2}\right]$$

Liu+Iqbal

- Exact in k_μ.
- First order non-linear differential equation.

Radial evolution of the response function

$$\Sigma(r) = -2\mathcal{A}_1(r, k_\mu)\sqrt{-\frac{g_{rr}}{g_{tt}}}$$

$$\Upsilon(r) = 2\mathcal{A}_0(r, k_\mu)\sqrt{-\frac{g_{tt}}{g_{rr}}}.$$

Boundary condition

- $$\boxed{\partial_r \bar{\chi}(k_\mu, r) = i\omega\sqrt{-\frac{g_{rr}}{g_{tt}}}\left[\frac{\bar{\chi}(k_\mu,r)^2}{\Sigma(r)} - \frac{\Upsilon(r)}{\omega^2}\right]}$$
- Boundary condition,

$$\bar{\chi}(k_\mu, r)^2\Big|_{r=1} = \frac{\Sigma(r)\Upsilon(r)}{\omega^2}\Big|_{r=1}.$$

- For two derivative gravity

$$\bar{\chi}(k_\mu, 1) = \sqrt{\frac{\Sigma(1)\Upsilon(1)}{\omega^2}} = \frac{1}{16\pi G_5}$$

1^{st} order transport coefficients from flow equation

- It is trivial to see that at $(\omega, k_i) \to 0$ limit,

$$\partial_r \bar{\chi}(k_\mu, r) = 0 \Rightarrow \bar{\chi}(k_\mu, r) = constant$$

- Using the boundary condition we get

$$\bar{\chi}(k_\mu, 1) = \bar{\chi}(k_\mu \to 0, \infty) = \eta$$

Flow from horizon to boundary

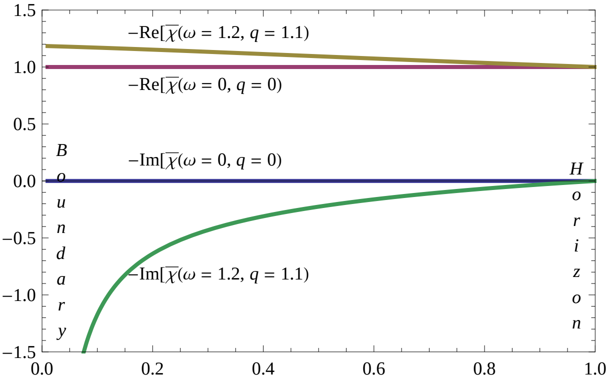

Figure: Flow of Green's function from horizon to boundary for two derivative gravity.

2^{nd} order transport coefficients from flow equation
NB, S. Dutta

$$\partial_r \bar{\chi}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[\frac{\bar{\chi}(k_\mu, r)^2}{\Sigma(r)} - \frac{\Upsilon(r)}{\omega^2} \right]$$

- To solve this equation up to order $\omega \rightarrow$ replace the leading value for $\bar{\chi}$

$$\partial_r \bar{\chi}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[\frac{\eta^2}{\Sigma(r)} - \frac{\Upsilon(r)}{\omega^2} \right] + \mathcal{O}(\omega^2, k_i^2).$$

- We impose the regularity condition at the horizon.

2^{nd} order transport coefficients from flow equation

✠ The solution

$$
\begin{aligned}
-i\omega\bar{\chi}(k_\mu,\infty) &= -i\omega\left(\frac{1}{16\pi G_5}\right) \\
&\quad +\omega^2\left[\frac{1}{2}(1-\ln 2)\left(\frac{1}{16\pi G_5}\right)\right] \\
&\quad -\frac{q^2}{2}\left(\frac{1}{16\pi G_5}\right)
\end{aligned}
$$

- Comparing it with the expression of Green's function

$$G_R^{xy,xy}(\omega,k) = -i\eta\omega + \eta\tau_\Pi\omega^2 - \tfrac{\kappa}{2}[(d-3)\omega^2+q^2]\,.$$

$$\eta = \frac{T^3\pi^3}{16\pi G_5}, \quad \kappa = \frac{\eta}{\pi T}, \quad \tau_\pi = \frac{2-\ln 2}{2\pi T}\,.$$

Observations

- Thus we see that the hydrodynamic characteristic of the field theory at UV fix point is completely determined in terms of this IR boundary condition.
- The flow equation may be visualized as RG equation.

Generalized flow equation in HD gravity

NB, S. Dutta

- We consider a gravity set-up with n derivative action.

$$\mathcal{I} = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left[R + 12 + \alpha' R^{(n)}\right]$$

- Not clear how to define the conjugate momentum and response function.
- Way out:

$$S_{eff} = \frac{1}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}dr\Big[\mathcal{A}_1^{HD}(r,k)\phi'(r,k)\phi'(r,-k) + \mathcal{A}_0^{HD}(r,k)\phi(r,k)\phi(r,-k)\Big]$$

NB, S. Dutta, JHEP 0903:116,2009

Generalized flow equation in HD gravity

§ Steps to write the flow equation in HD gravity

- Generalized momentum

$$\Pi^{HD}(r,k) = 2\mathcal{A}_1^{HD}(r,k)\phi'(r,k)$$

- Boundary Green's function

$$\begin{aligned} G_R^{HD}(k_\mu) &= \lim_{r\to\infty} -\frac{2\mathcal{A}_1^{HD}(r,k)\phi'(r,k)\phi(r,-k)}{\phi_0(k)\phi_0(-k)} \\ &= \lim_{r\to\infty} -\frac{\Pi^{\mathrm{HD}}(r,k_\mu)}{\phi(r,k_\mu)} \end{aligned}$$

- Define a response function in higher derivative theory

$$\bar{\chi}^{HD}(k_\mu, r) = \frac{\Pi^{HD}(r,k_\mu)}{i\omega\phi(r,k_\mu)}.$$

★ Therefore the flow equation is given by,

$$\partial_r \bar{\chi}^{HD}(k_\mu, r) = i\omega\sqrt{-\frac{g_{rr}}{g_{tt}}}\left[\frac{\bar{\chi}^{HD}(k_\mu, r)^2}{\Sigma^{HD}(r,k)} - \frac{\Upsilon^{HD}(r,k)}{\omega^2}\right]$$

Here we define

$$\begin{aligned} \Sigma^{HD}(r,k) &= -2\mathcal{A}_1^{HD}(r,k_\mu)\sqrt{-\frac{g_{rr}}{g_{tt}}} \\ \Upsilon^{HD}(r,k) &= 2\mathcal{A}_0^{HD}(r,k_\mu)\sqrt{-\frac{g_{tt}}{g_{rr}}}. \end{aligned}$$

Boundary Condition

- The response function $\bar{\chi}^{HD}$ should be well-defined at horizon. This implies,

$$\bar{\chi}^{HD}(k_\mu, r)\Big|_{r=r_h} = \sqrt{\frac{\Sigma^{HD}(r)\Upsilon^{HD}(r)}{\omega^2}}\Bigg|_{r=r_h}$$

Flow from horizon to boundary in HD gravity

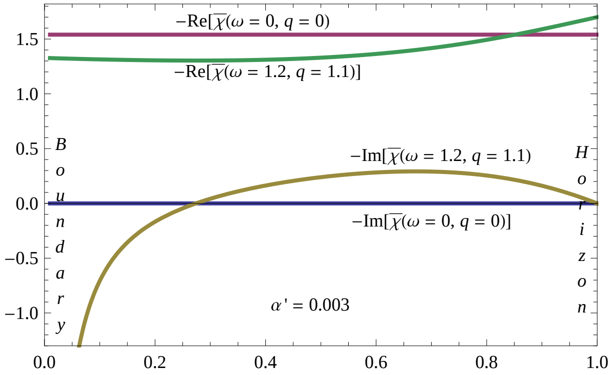

Figure: Flow of Green's function from horizon to boundary in higher derivative gravity.

Results

- Exact Gauss-Bonnet theory

$$
\begin{aligned}
\eta &= \frac{1}{16\pi G_5}(1-4\lambda_{gb}) \\
\kappa &= \frac{2\lambda_{gb}\,(8\lambda_{gb}-1)}{\left(1-\sqrt{1-4\lambda_{gb}}\right)(4\lambda_{gb}-1)} \\
\tau_\pi T &= \frac{1}{4\pi(-1+4\lambda_{gb})}\Big[-8\lambda_{gb}^2+12\sqrt{1-4\lambda_{gb}}\lambda_{gb} \\
&+10\lambda_{gb}-2\sqrt{1-4\lambda_{gb}}-4\log(2)\lambda_{gb} \\
&+(1-4\lambda_{gb})\log\left(-4\lambda_{gb}+\sqrt{1-4\lambda_{gb}}+1\right) \\
&+(4\lambda_{gb}-1)\log(1-4\lambda_{gb})-2+\log(2)\Big].
\end{aligned}
$$

Results

- To preserve causality of a conformal fluid there exists a bound [Buchel+Myers]

$$\boxed{\tau_\pi T - 2\frac{\eta}{s} \geq 0 \ .}$$

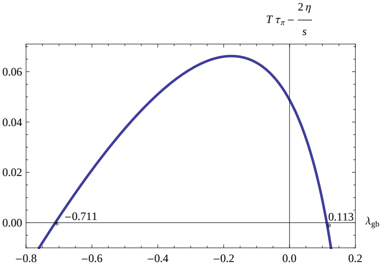

Figure: Bound on λ_{gb}

- The action

$$S_{\mathrm{EM}} = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(R + 12 - F^2\right) + S_{\mathrm{CS}}$$

Kraus + Dhoker

- Three possible extremal near-horizon geometry.
-

$$qB(\zeta^2 - 1) = 0$$

- Shear viscosity to entropy density

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

Magnetic Brane Solution

- The higher derivative term

$$S_{\mathrm{HD}} = \frac{\alpha'}{16\pi G_5} \int d^5x \sqrt{-g} \Big[\mathrm{c_1 Rim^2 + c_2 Rim \cdot F^2 + c_3 (F^2)^2} + c_4\, F^4 + c_5\, \mathrm{GCS} \Big]$$

Myers + paulos + Sinha

- Shear viscosity to entropy density

$$\frac{\eta}{s} = \frac{1}{4\pi} + \frac{\alpha'}{\pi}\left[\frac{c_1}{3}\left((q^2 - \frac{B^2}{r_h^4}) - 6\right) + \frac{c_2}{2}(q^2 - \frac{B^2}{r_h^4})\right] + \mathcal{O}(\alpha'^2)$$

Magnetic Brane Solution

- Second order

$$\kappa = \frac{\eta}{\pi T}\left[\left(1-\frac{q^2}{2r_0^6}\right)-\alpha'\left(10c_1-\frac{q^2}{3r_0^6}(37c_1-48c_2)\right)\right]$$

$$\tau_\pi T = \frac{2-\ln 2}{2\pi}-\frac{q^2(5-3\ln 2)}{4\pi r_0^6}$$
$$+\alpha'\left[-\frac{11c_1}{2\pi}+\frac{q^2}{4\pi r_0^6}(-16c_2+5c_1(11-4\ln 2))\right]$$

- We study the flow of retarded Green's function of energy momentum tensor in presence of generic matter coupling and higher derivative gravity.
- The flow equation is valid for any momentum: $k_\mu \rightarrow$ Any higher order transport coefficient can be computed from this flow equation.
- It would be interesting to understand the connection between low frequency behavior of membrane fluid and boundary fluid using this flow equation. *Any UV/IR relation*??

Thank you