Gravity duals of $\mathcal{N} = 2$ superconformal field theories with no electrostatic description

K. SIAMPOS,
Mécanique et Gravitation,
Université de Mons

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with P. M. Petropoulos and K. Sfetsos

Crete Center for Theoretical Physics

Plan of the talk

1. Gravity duals of $\mathcal{N} = 2$ SCFTs
2. Methods for solving Toda
   - Known 11d solutions so far – Electrostatics
   - Comments on electrostatics
   - Beyond electrostatics
3. Our solution
   - Gravitational instantons in 4d as a tool
   - Bianchi IX foliations and self–duality
   - Toda frame of the Atiyah–Hitchin metric
   - Appropriate boundary conditions
4. In progress
5. Discussion & Outlook

SYNOPSIS

We constructed the first 11d supergravity solutions with $SO(2,4) \times SO(3) \times U(1)_R$ isometry, which are regular and have no smearing.

Absence of an extra $U(1)$ symmetry, even asymptotically – No electrostatic description

"Short motivation": Explore the 11d landscape of qualitatively different solutions and potentially understand the dual SCFTs.

Gravity duals of $\mathcal{N} = 2$ SCFTs

Solutions of 11d SUGRA which possess $SO(2,4) \times SO(3) \times U(1)_R$ isometry were constructed in Lin–Lunin–Maldacena (2004):

$$
\begin{aligned}
& \mathrm{d}s^2_{11} = \kappa_{11}^{\frac{2}{3}} e^{2\lambda} \left(4\, \mathrm{d}s^2_{\mathrm{AdS}_5} + z^2 e^{-6\lambda} \mathrm{d}\Omega_2^2 + \frac{4}{1 - z\,\partial_z \Psi} (\mathrm{d}\varphi + \omega)^2 - \frac{\partial_z \Psi}{z} \gamma_{ij} \mathrm{d}x^i \mathrm{d}x^j \right) , \\
& \omega = \omega_x \mathrm{d}x + \omega_y \mathrm{d}y \,, \quad \omega_x = \frac{1}{2} \partial_y \Psi \,, \quad \omega_y = -\frac{1}{2} \partial_x \Psi \,, \\
& \gamma_{ij} \mathrm{d}x^i \mathrm{d}x^j = \mathrm{d}z^2 + e^{\Psi} (\mathrm{d}x^2 + \mathrm{d}y^2) \,, \quad e^{-6\lambda} = -\frac{\partial_z \Psi}{z(1 - z\,\partial_z \Psi)} \,, \quad G_4 = \mathrm{d}C_3 = \kappa_{11} F_2 \wedge \mathrm{d}\Omega_2 \,, \\
& F_2 = 2(\mathrm{d}\varphi + \omega) \wedge \mathrm{d}\left(z^3 e^{-6\lambda}\right) + 2z\left(1 - z^2 e^{-6\lambda}\right) \mathrm{d}\omega - \partial_z e^{\Psi} \mathrm{d}x \wedge \mathrm{d}y \,.
\end{aligned}
$$

where $\Psi(x,y,z)$ satisfies the continual Toda equation [continuum Lie algebras – Saveliev (1990)]:

$$
\left(\partial_x^2 + \partial_y^2\right) \Psi + \partial_z^2 e^{\Psi} = 0 \,,
$$

where $z \in [0, z_c]$ and $z_c : e^{\Psi} = 0$. The boundary conditions for the 11d background regularity:

$$
z = 0 : \quad e^{\Psi} = \text{finite} \neq 0 \,, \quad \partial_z \Psi = 0 \,, \quad \partial_z \Psi / z = \text{finite.}
$$

Only known regular solutions so far involve separability or existence of an extra $U(1)$ isometry.

Methods for solving Toda

Separable solutions

They boil down to the Liouville equation

$$e^{\Psi} = c_3 \frac{|\partial f|^2}{(1 - c_3 |f|^2)^2} \left(-z^2 + c_1 z + c_2\right) , \quad q = \frac{1}{2}(x + iy).$$

where c_i's are constants and $f = f(q)$ is a locally univalent meromorphic function.
Example: Maldacena–Núñez

$$e^{\Psi} = 4 \frac{N^2 - z^2}{(1 - r^2)^2} , \quad z \in [0, N] , \quad r \in [0, 1] .$$

An extra $U(1)$ symmetry

- The problem can be mapped to a Laplace equation – electrostatics Ward (1990).
- **Examples:** Maldacena–Nũnéz and

  $$AdS_7 \times S^4 : \quad e^{\Psi} = \coth^2 \zeta , \quad r = \sinh^2 \zeta \sin \vartheta , \quad z = \cosh^2 \zeta \cos \vartheta , \quad \zeta \in \mathbb{R}^*_+ , \quad \vartheta \in [0, \pi/2] .$$

Known 11d solutions so far – Electrostatics

Ward's transformation:

$$\ln r = \partial_\eta \Phi\,, \quad z = \rho \partial_\rho \Phi\,, \quad \rho = r\,\mathrm{e}^{\Psi(r,z)/2}\,,$$

maps the Toda equation to a Laplace equation in cylindrical coordinates (ρ, η)

$$\frac{1}{r}\partial_r\left(r\partial_r\Psi\right) + \partial_z^2 e^{\Psi} = 0 \Longrightarrow \frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho\Phi\right) + \partial_\eta^2\Phi = 0\,,$$

on an infinite conducting plane ($\vec{E} = \vec{E}_\perp$) with a charge density $\lambda(\eta)$ along positive half-axis of η.

Examples of line charge densities:

- Maldacena–Nũnéz

$$\lambda(\eta) = \begin{cases} \eta\,, & 0 \leqslant \eta \leqslant N\,, \\ N\,, & \eta \geqslant N\,. \end{cases}$$

- $\mathrm{AdS}_7 \times S^4$

$$\lambda(\eta) = \begin{cases} 2\eta\,, & 0 \leqslant \eta \leqslant \frac{1}{2}\,, \\ \eta + \frac{1}{2}\,, & \eta \geqslant \frac{1}{2}\,. \end{cases}$$

The line charge distribution over an infinite plane has an one to one correspondence with $\mathcal{N} = 2$ quiver gauge theories Gaiotto–Maldacena (2012).

COMMENTS ON ELECTROSTATICS

Line charge density – related to the M5 sources

Gaiotto–Maldacena, Reid-Edwards–Stefanski (2011), Donos–Simon (2011) & Aharony–Berdichevsky–Berkooz (2012)

- Extra $U(1)$ isometry endows a smearing process with the typical validity limitations. An exception is the Maldacena–Ñũnéz solution, i.e. no smearing and regular punctures.
- Regularity of spacetime imposes constraints on $\lambda(\eta)$, arising from 4-flux quantisation on punctures.
  - $\lambda(\eta)$ is continuous and piecewise segment, i.e. $a_n\eta + q_n$, where $a_n \in \mathbb{Z}$.
  - Kinks occur at integer values of η.
  - $\lambda(0) = 0$ and $a_{n-1} - a_n = k_n \in \mathbb{Z}_+ \Longrightarrow A_{k_n-1}$ singularity transverse to $AdS_5 \times S^2$.

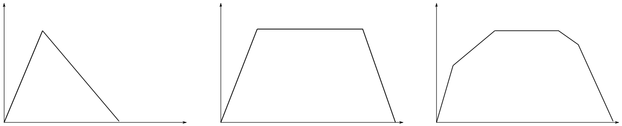
A class of 4d $\mathcal{N} = 2$ SCFTs can be viewed as generalised quiver gauge theories

Gaiotto (2009)

- $\exists\, SU(\lambda_n)$ gauge group $\forall\lambda_n = \lambda(\eta)_{\eta=n} : \lambda_n \leqslant \lambda_{n+1} \leqslant N \Longrightarrow k_n := 2\lambda_n - \lambda_{n-1} - \lambda_{n+1} < 0\,.$
- $\forall\,\mathrm{Kink}_{\eta=n}$, $\exists k_n = a_{n-1} - a_n$ fundamental hypermultiplets charged under the $SU(\lambda_n) \Longrightarrow A_{k_n-1}$.
- In total, this is a quiver with gauge group $\prod_n SU(n)$ described at strong coupling by supergravity.

Beyond electrostatics

Charge distributions with A_{k-1} singularities – irregular punctures; $\mathbb{R}^4/\mathbb{Z}_k$



For $k \neq 1$ and large values of ρ: Electrostatics picture $\Longrightarrow$ a non-$U(1)$ Toda solution.

Objective: Find Toda potentials which are not separable and depend on x, y, z.

General solutions of the continual Toda equation are not known.

Genuine solutions have been found in the framework of triaxial Bianchi-IX four-dimensional instantons.

- Riemann self-dual by Atiyah–Hitchin (1985).
- Kähler and $R = 0$ (WASD) by Pedersen–Poon (1990).
- Weyl self-dual and Einstein by Tod (1994) & Hitchin (1995).

We move on with a revisit on gravitational instantons in 4d and the Atiyah–Hitchin solution.

Gravitational instantons in 4d as a tool

HyperKähler manifold, with a Killing vector $\xi = \partial_\varphi$

$$\mathrm{d}\ell^2 = V\,\left(\mathrm{d}\varphi + \omega_i\,\mathrm{d}x^i\right)^2 + V^{-1}\,\mathrm{d}s^2\,,\quad \mathrm{d}s^2 = \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,,\quad i = 1, 2, 3$$

Gibbons–Hawking (1979)

Self-duality of Riemann tensor yields two district types of Killing vectors: *Translational* and *Rotational*.

Boyer–Finley (1982) & Gegenberg–Das (1984)

1. *Translational* Killing vector

$$\mathrm{d}V^{-1} = \pm \star_\gamma\,\mathrm{d}\omega\,,\quad \gamma_{ij} = \delta_{ij}\,,\quad \partial_i\partial^i V^{-1} = 0\,,\quad V^{-1} = \varepsilon + \sum_{i=1}^{n} \frac{m_i}{|\vec{x} - \vec{x}_{0i}|}\,.$$

2. *Rotational* Killing vector – Toda frame

$$\mathrm{d}s^2 = \mathrm{d}z^2 + \mathrm{e}^{\Psi}(\mathrm{d}x^2 + \mathrm{d}y^2)\,,\quad V^{-1} = \frac{1}{2}\partial_z\Psi\,,\quad \omega_x = \frac{1}{2}\partial_y\Psi\,,\quad \omega_y = -\frac{1}{2}\partial_x\Psi\,,$$

Self-duality of the Riemann tensor yields the continual Toda equation.

Regularity requirement of the 4d (tool) geometry is opposite to the 11d one

$$R_{\kappa\lambda\mu\nu}\,R^{\kappa\lambda\mu\nu} \sim (\partial_z\Psi)^{-6}\,.$$

BIANCHI IX FOLIATIONS AND SELF–DUALITY

Invariant metric under the left-action of the $SU(2)$ algebra ξ_i (right-invariant) fields

$$\mathrm{d}\ell^2 = \frac{1}{4}\,\Omega_1\Omega_2\Omega_3\,\mathrm{d}t^2 + \frac{\Omega_2\Omega_3}{\Omega_1}\,\sigma_1^2 + \frac{\Omega_1\Omega_3}{\Omega_2}\,\sigma_2^2 + \frac{\Omega_1\Omega_2}{\Omega_3}\,\sigma_3^2\,,$$

$$\sigma_1 + i\sigma_2 = -\mathrm{e}^{i\,\psi}\,(i\,\mathrm{d}\vartheta + \sin\vartheta\,\mathrm{d}\varphi)\,, \quad \sigma_3 = \mathrm{d}\psi + \cos\vartheta\,\mathrm{d}\varphi\,, \quad \mathrm{d}\sigma_i = \frac{1}{2}\,\varepsilon_{ijk}\sigma_j\wedge\sigma_k\,.$$

Self-duality yields the 1^{st} order differential system

$$\dot{\Omega}_1 = \frac{\mathrm{d}\Omega_1(t)}{\mathrm{d}t} = \frac{1}{2}\left(\Omega_2\Omega_3 - \lambda\,\Omega_1\,(\Omega_2+\Omega_3)\right)\,, \quad \text{and cyclic}\,.$$

$\lambda = 0$ – ξ_i translational – Lagrange system – Algebraically integrable

Axisymmetric – Triaxial solutions found with a few months interval by
Eguchi–Hanson (1978) & Belinsky–Gibbons–Page–Pope (1978).

$\lambda = 1$ – ξ_i rotational – Darboux–Halphen system – Not always algebraically integrable

The axisymmetric is the Taub–NUT, known since the 50s (Lorentzian) but revived as an instanton of SU(2) foliations by Gibbons–Hawking (1978) & Eguchi–Hanson (1979).
The triaxial was found by: Atiyah–Hitchin (1985).

General solution was known since the 19th century by Halphen's (1881) works.

Toda frame of the Atiyah–Hitchin metric

The Toda frame was found by Olivier (1991) for the Atiyah–Hitchin metric and generalised by Finley–McIver (2010) for the general solution of the Halphen system

$$z=\frac{1}{2}\sum_{i=1}^{3}\Omega_i\left(1-n_i^2\right)\,,\quad q=\frac{1}{2}\,(x+i\,y)=q(\Omega_i,\vartheta,\psi\text{ via elliptic integrals})\,,$$

$$e^{2\Psi}=4\sum_{i,j=1}^{3}\Omega_i\Omega_j\left(n_i^2+n_j^2+n_i^2n_j^2-1-2(2n_in_j-1)\delta_{ij}\right)\,,\quad V=\sum_{i=1}^{3}n_i^2\,\frac{\Omega_j\Omega_k}{\Omega_i}\,,$$

$$\text{where :}\qquad n_i=(\cos\psi\,\sin\vartheta,\sin\psi\,\sin\vartheta,\cos\vartheta)\,.$$

We shall apply the b.c. for the 11d background regularity on Ω_i and $x^i=(t,\vartheta,\psi)$ parameters.

Appropriate boundary conditions

Boundary conditions for 11d regularity are equivalent to:

$$\forall(\vartheta^*,\psi^*) \Longrightarrow \exists\, t^* : \; n_k^* \neq 0\,, \quad \Omega_k^* = 0\,, \quad \alpha\,\Omega_i^* + \beta\,\Omega_j^* = 0\,, \quad \alpha+\beta = 2\,, \quad \alpha,\beta \in (0,2]\,,$$

where (α, β) are expressed in terms of $n_i^* = n_i(\vartheta^*,\psi^*)$.

They possess a smooth limit as $z \longrightarrow 0$

$$\frac{\partial_z \Psi}{z} = \frac{2}{z\,V} = -\frac{2}{\Omega_i^*\,\Omega_j^*\,n_k^{*4}} = \text{finite} \neq 0\,.$$

Comments on the solution:

- It is a genuine triaxial case.
- There are no punctures, i.e. $e^{\Psi} \neq 0$ – no additional non-compact branes.
- The axisymmetric limit (Taub–NUT) does not satisfy the b.c. – No electrostatic description.

Simple poles of Ω_i – Singular four-dimensional instantons – Regular eleven-dimensional solutions.

THE HALPHEN SOLUTION

The original Halphen (Atiyah–Hitchin) regular solution $\Omega_{i,H}(T)$, in terms of Theta functions and the quasimodular form of weight two or complete elliptic integrals, reads:

$$\Omega_{1,H} = \frac{\pi}{6}\left(E_2(iT) - \theta_2^4(iT) - \theta_3^4(iT)\right) = -K(\kappa)(E(\kappa) - \kappa'^2 K(\kappa)),$$
$$\Omega_{2,H} = \frac{\pi}{6}\left(E_2(iT) + \theta_3^4(iT) + \theta_4^4(iT)\right) = -K(\kappa)E(\kappa),$$
$$\Omega_{3,H} = \frac{\pi}{6}\left(E_2(iT) + \theta_2^4(iT) - \theta_4^4(iT)\right) = -K(\kappa)(E(\kappa) - K(\kappa)),$$

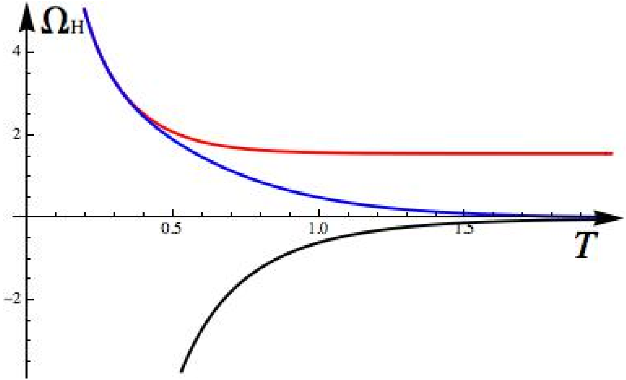
Where κ is the elliptic modulus

$$\frac{t}{2} = T = -\frac{2K(\kappa')}{\pi K(\kappa)}, \quad \kappa'^2 = 1 - \kappa^2.$$

This solution describes:

- A regular SU(2)-symmetric self-dual gravitational instanton.
- The configuration space of two slowly moving BPS $SU(2)$ Yang–Mills–Higgs monopoles. Manton (1982) & Gibbons–Manton (1986)

The Halphen solution



Halphen original solution $\Omega_{1,H} < 0 < \Omega_{3,H} < \Omega_{2,H}$.

Appropriate boundary conditions

The $SL(2,\mathbb{R})$ covariance of the Darboux–Halphen system

$$\Omega_i(T) = \frac{1}{(CT+D)^2}\Omega_{i,H}\left(\frac{AT+B}{CT+D}\right) + \frac{C}{CT+D}\,, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{R})\,, \quad T = \frac{t}{2}\,.$$

Generic $SL(2,\mathbb{R})$ transformation, yield a singular 4d geometry.

Note that z transforms like Ω_i and the Toda potential as

$$\mathrm{e}^{2\Psi(T,\vartheta,\psi)} = \frac{1}{(CT+D)^4}\,\mathrm{e}^{2\Psi_H\left(\frac{AT+B}{CT+D},\vartheta,\psi\right)}\,.$$

Satisfying the boundary conditions

We can satisfy the b.c. ($\Omega_3^* = 0$ & $\alpha\Omega_1^* + \beta\Omega_2^* = 0$) if

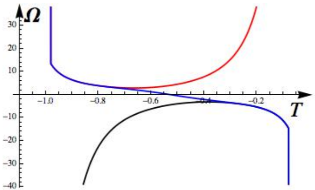
$$T_* = \frac{1}{\pi C^2}\left[\alpha\,K\left(\sqrt{\beta/2}\right) - 2\,E\left(\sqrt{\beta/2}\right)\right]K\left(\sqrt{\beta/2}\right) - \frac{D}{C}\,,$$

$$\frac{A}{C} = -\frac{\beta\,K\left(\sqrt{\alpha/2}\right) - 2\,E\left(\sqrt{\alpha/2}\right)}{\alpha\,K\left(\sqrt{\beta/2}\right) - 2\,E\left(\sqrt{\beta/2}\right)}\,.$$

where

$$T_0 < T < T_\infty\,, \quad T_0 = -\frac{B}{A}\,, \quad T_\infty = -\frac{D}{C}\,.$$

Appropriate boundary conditions



Generic solution in the range $T_0 < T < T_\infty$ and $\Omega_1 < \Omega_3 < \Omega_2$.

Given value of α, there is a 2-parameter family due to unimodularity.

Metric signature demands $\frac{\partial_z \Psi}{z} < 0 \Longrightarrow T \in [T_0, T_*]$.

For the symmetric case, $\alpha = \beta = 1 \Longrightarrow T_* = \frac{1}{2}\left(T_0 + T_\infty\right)$.

The Atiyah–Hitchin (non-singular) solution corresponds to $T < T_\infty < T_0 < T$ and does not satisfy the b.c.

IN PROGRESS

Kähler metric with symmetry and $R = 0$ – WASD (canonical orientation)

LeBrun (1991)

$$\text{Once more :} \quad d\ell^2 = V(d\varphi + A)^2 + V^{-1}(dz^2 + e^{\Psi}(dx^2 + dy^2))\,,$$

where the $R = 0$ and Kähler conditions

$$\left(\partial_x^2 + \partial_y^2\right)\Psi + \partial_z^2\left(e^{\Psi}\right) = e^{\Psi}\nabla^2\Psi = 0\,,$$

$$\left(\partial_x^2 + \partial_y^2\right)V^{-1} + \partial_z^2\left(V^{-1}\,e^{\Psi}\right) = 0\,,$$

$$A = \partial_x V^{-1} dy \wedge dz + \partial_y V^{-1} dz \wedge dx + \partial_z\left(V^{-1}\,e^{\Psi}\right) dx \wedge dy\,,$$

with Kähler form

$$J = (d\varphi + A) \wedge dz - V^{-1}\,e^{\Psi} dx \wedge dy\,, \quad dJ = 0\,.$$

Application: Non-supersymmetric 5d multi-center solutions – LeBrun metrics as base space.

Bobev–Niehoff–Warner (2011), (2012) & Niehoff (2013)

Known solutions involve methods of electrostatics.

PEDERSEN–POON METRICS

General diagonal Kähler and $R=0$ (WASD) metric with $SU(2)$ isometry

Pedersen–Poon (1990)

$$\Omega_1' = \Omega_2\Omega_3 - \alpha\Omega_1\,, \quad \Omega_2' = \Omega_1\Omega_3 - \alpha\Omega_2\,, \quad \Omega_3' = \Omega_1\Omega_2\,, \quad T = \frac{t}{2}\,.$$

where α is a constant. For $\alpha = 0$ – BGPP metric, i.e. RSD – ∂_φ-translational.

Its LeBrun frame

Tod (1995)

$$z = n_3\,\Omega_3\,, \quad x = e^{\alpha T}\,n_2\,\Omega_2\,, \quad y = e^{\alpha T}\,n_1\,\Omega_1\,, \quad e^{\Psi} = e^{-2\alpha T}\,,$$

$$V = \sum_{i=1}^{3}\frac{\Omega_j\Omega_k}{\Omega_i}\,n_i^2\,, \quad A_i\,\mathrm{d}x^i = V^{-1}\left(\left(\frac{\Omega_1\Omega_3}{\Omega_2} - \frac{\Omega_2\Omega_3}{\Omega_1}\right)\sin\vartheta\,\sin\psi\,\cos\psi\,\mathrm{d}\vartheta + \frac{\Omega_1\Omega_2}{\Omega_3}\,\cos\vartheta\,\mathrm{d}\psi\right)\,,$$

An application in 11d SUGRA:

- Triaxial solutions – regular 11d supegravities with $SO(2,4)\times SO(3)\times U(1)_R$.
- There is a regular puncture at $e^{\Psi}=0, \vartheta=0 \Longrightarrow \alpha = N_5$.
- $\exists$ axisymmetric solutions – extra $U(1)$ isometry – usual electrostatics-quiver description: $\alpha\in\mathbb{N}$.

Discussion & Outlook

- We constructed the first 11d supergravity solutions, which are regular, have no smearing and possess only $SO(2,4) \times SO(3) \times U(1)_R$ isometry.
- Use of an auxiliary problem: We retrieved the Toda potential for metrics with $SU(2)$ rotational symmetry.
- These are obtained for example by transforming the Atiyah–Hitchin instanton under $SL(2,\mathbb{R})$.
- Absence of an extra $U(1)$ symmetry, even asymptotically – no electrostatic description.
- Qualitatively different class of solutions dual to new 4d $\mathcal{N}=2$ SCFTs. Our next goal is to unravel them...

Other applications of the instantons tools:

1. Non-supersymmetric multi-center solutions – LeBrun metrics as base space.
   Bobev–Niehoff–Warner (2011) & (2012) & Niehoff (2013)
2. Scalar hypermultiplets in $\mathcal{N}=2$ supergravities – 4d quaternionic spaces.
   Bagger–Witten (1983)