Null surface geometry, fluid vorticity, and turbulence

Christopher Eling[1]

[1] MPI for Gravitational Physics (Albert Einstein Institute), Potsdam

February 25, 2014

Outline

- Introduction and Background: Holography and fluids
  - Hydrodynamics (relativistic CFT and the non-relativistic limit)
  - Fluid-gravity correspondence
- Null surface dynamics (Eling, Fouxon, Neiman, Oz 2009-2011)
  - Null Gauss-Codazzi equations encode boundary fluid dynamics
  - Fluid vorticity $\rightarrow$ horizon "rotation two-form" (Eling and Oz, 1308.1651)
- A Geometrization of turbulence
  - For 4d black brane dual to 2+1 d fluid, vorticity scalar mapped to Ψ_2 Newman-Penrose scalar
  - Statistical scaling of horizon structure
- Discussion

AdS/CFT and Hydrodynamics

- Holographic principle: microscopic gravity dof in a volume V encoded on a boundary A of region
- Concrete realization of holographic principle in AdS/CFT (or more generally gauge/gravity) correspondences
  - Quantum gravity is equivalent to some gauge theory in one lower dimension “on the boundary"
- Most studied regime: where the bulk theory is classical gravity and the dual gauge theory is (infinitely) strongly coupled
- A thermal state of the gauge theory $\Leftrightarrow$ a classical black hole spacetime
- Consider long wavelength, long time perturbations of the BH $\Leftrightarrow$ Hydrodynamics of the gauge theory (Policastro, Son, and Starinets 2001)

(Relativistic) Hydrodynamics

- Universal description of large scale (long time, wavelength) dynamics of a field theory
- Regime where the *Knudsen* number $\epsilon \equiv \frac{\ell_{corr}}{L} \ll 1$
- Microscopic theory obeys exact conservation laws, e.g.

$$\partial_\nu T^{\mu\nu} = 0, \tag{1}$$

$$\rho = T^{00}, \Pi^i = T^{0i} \tag{2}$$

- Constitutive relation: Kn (gradient) expansion

$$T^{ij} = P(\rho)\delta^{ij} + \partial^i \Pi^j + \partial^2 + \cdots \tag{3}$$

- Viscous stress tensor

$$T_{\mu\nu} = (\rho + P)u^\mu u^\nu + P\eta^{\mu\nu} - 2\eta\sigma^{\mu\nu} - \zeta(\partial \cdot u)P^{\mu\nu} + \cdots \tag{4}$$

  - η shear viscosity, ζ bulk viscosity, $P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$, $\sigma_{\mu\nu} = P^\lambda_\mu P^\sigma_\nu \partial_{(\lambda} u_{\sigma)}$

CFT Hydrodynamics in $d+1$ dimensions

- Traceless stress tensor $T^\mu_\mu = 0$

$$T^{\mu\nu} \sim T^{d+1}\left(\eta^{\mu\nu} + (d+1)u^\mu u^\nu\right) - 2\eta\sigma^{\mu\nu} \tag{5}$$

- Projected Equations at Ideal order (neglect viscous pieces)

$$P_{\nu\sigma}\partial_\mu T^{\mu\sigma} = \Omega_{\mu\nu}u^\nu = 0, \tag{6}$$
$$u_\nu\partial_\mu T^{\mu\nu} = \partial_\mu s^\mu = 0 \tag{7}$$

- Conserved entropy current, relativistic enstrophy two-form

$$s^\mu = T^d u^\mu, \quad \Omega_{\mu\nu} = \partial_{[\mu}(Tu_{\nu]}) \tag{8}$$

Non-relativistic limit

- $u^{\mu} = \gamma(1, v^i/c)$, $v \ll c$

$$\partial_i \sim \lambda, \partial_t \sim \lambda^2, v^i \sim \lambda, T = T_0(1 + \lambda^2 p(x)) \tag{9}$$

  $\lambda \sim c^{-1}$

- Fouxon and Oz 2008; Bhattacharyya, Minwalla, Wadia 2008
- Incompressible Euler equations of everyday flows

$$\partial_i v^i = 0 \tag{10}$$

$$\partial_t v_i + v^j \partial_j v_i + \partial_i p = 0. \tag{11}$$

Fluid/gravity correspondence

- Idea: Black hole geometry dual to an ideal fluid (on flat spacetime) at temperature T in global equilibrium
- To make manifest: write black brane metric in boosted form (Bhattacharyya, Hubeny, Minwalla, Rangamani 2008)

$$ds^2 = -F(r)u_\mu u_\nu dx^\mu dx^\nu - 2u_\mu dx^\mu dr + G(r)P_{\mu\nu}dx^\mu dx^\nu, \qquad (12)$$

$x^A = (r, x^\mu)$; $u^\mu = \gamma(1, v^i)$, $F(0) = 0$ the horizon
Entropy: $s = v/4 = G(0)/4$, Hawking temperature
$T = \kappa/2\pi = -F'(0)/2$

- Particular class of metrics: AdS black branes

$$R_{AB} + dg_{AB} = 0 \qquad (13)$$

Perturbing the metric

- Now let $u^\mu(x^\mu)$ and $T(x^\mu)$- similar to “variation of constants" in Boltzmann equation in kinetic theory

$$ds^2_{(0)} = -F(r,x^\mu)u_\mu(x)u_\nu(x)dx^\mu dx^\nu - 2u_\mu(x)dx^\mu dr + G(r,x^\mu)P_{\mu\nu}(x)dx^\mu dx^\nu \quad (14)$$

- Expand approximate bulk gravity solution order by order in Knudsen number. Expansion in parameter ϵ counts derivatives of u^μ, T, etc
- Solve order by order in ϵ starting with the equilibrium metric (local equilibrium)

Constraint equations and boundary stress tensor

- The GR momentum constraint equations on “initial" data at the AdS boundary are the Navier-Stokes equations for a fluid

$$G_A^{(n)\nu} N^A = \partial_\mu T^{\mu\nu}_{BY(n)} = 0 \tag{15}$$

N^A unit spacelike normal

- $T^{\mu\nu}_{BY}$ is the quasi-local Brown-York stress tensor at the boundary

$$T^{BY}_{\mu\nu} = \frac{1}{8\pi G}(K\gamma_{\mu\nu} - K_{\mu\nu} + counterterms) \tag{16}$$

$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_N \gamma_{\mu\nu}$

- Computation for metric $g^{(0)}_{\mu\nu}$ reveals this is exactly the ideal fluid stress tensor. Conservation = relativistic Euler eqns

Horizon geometry

- Past work (Eling and Oz 2009) we showed one can express Gauss-Codazzi equations for the horizon (plus field eqns) as the hydro equations
- Choose coordinates so that $r = 0$ is horizon. Null normal is

$$\ell^A = g^{AB}\nabla_B r = (0, \ell^\mu) \quad (17)$$

- Induced metric $\gamma_{\mu\nu}$ is pullback of g_{AB} to horizon. It is *degenerate*: $\gamma_{\mu\nu}\ell^\nu = 0$
- Second fundamental form

$$\theta_{\mu\nu} \equiv \frac{1}{2}\mathcal{L}_\ell \gamma_{\mu\nu} = \sigma^{(H)}_{\mu\nu} + \frac{1}{d-1}\theta\gamma_{\mu\nu} \quad (18)$$

- Horizon expansion in terms of area entropy current $S^\mu = v\ell^\mu$

$$\theta = v^{-1}\partial_\mu(v\ell^\mu) = v^{-1}\partial_\mu S^\mu \quad (19)$$

Horizon dynamics

- “Weingarten map":

$$\nabla_{\mu}\ell^{\nu} = \Theta_{\mu}{}^{\nu} = \theta_{\mu}{}^{\nu} + c_{\mu}\ell^{\nu}; \quad c_{\mu}\ell^{\mu} = \kappa \,. \tag{20}$$

- c_{μ} horizon's “rotation one-form" (in GR literature) - encodes temperature and velocity
- We showed Null Gauss-Codazzi equations have form Eling, Neiman, Oz 2010

$$R_{\mu\nu}S^{\mu} = c_{\mu}\partial_{\nu}S^{\mu} + 2S^{\nu}\partial_{[\nu}c_{\mu]} + F(\theta, \sigma^{(H)}_{\mu\nu}) = 0 \tag{21}$$

- For the black brane metric above, at lowest order in derivatives

$$S^\mu = 4su^\mu; \quad \Theta_{\mu\nu} = -2\pi T u_\mu u^\nu; \quad \gamma_{\mu\nu} = (4s)P_{\mu\nu}; \quad c_\mu = -2\pi T u_\mu \tag{22}$$

- Conservation of Area current– *a non-expanding horizon*

$$\partial_\mu S^\mu = \theta = 0; \quad \Omega_{\mu\nu} \sim \partial_{[\nu} c_{\mu]} \tag{23}$$

- Non-relativistic limit, Euler equation

$$\theta = 0 \rightarrow \partial_i v^i = 0 \tag{24}$$

Viscous corrections

- Can also get viscous corrections from null focusing (Raychaudhuri) equation, e.g.

$$\partial_\mu (s\ell^\mu) = \frac{1}{4}\partial_\mu S^\mu = \frac{s}{2\pi T}\sigma_{\mu\nu}\sigma^{\mu\nu} \tag{25}$$

- Recover shear viscosity to entropy density ratio $\eta/s = 1/4\pi$
- Non-relativistic limit and Second Law

$$\partial_i v^i \sim \nu \int \partial_i v_j \partial^i v^j d^d x \tag{26}$$

$$\partial_t A \sim -\int \partial_t \frac{1}{2} v^2 d^d x \tag{27}$$

- What does geometrization imply about turbulent flows?

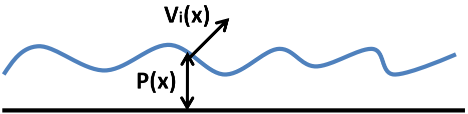


Figure 3: fluid pressure and velocity in the geometrical picture. The pressure P(x) measures the deviation of the perturbed event horizon from the equilibrium solution. The velocity vector field Vi(x) is the normal vector.

Turbulent flows

$$\partial_t v_i + v^j \partial_j v_i + \partial_i p = \nu \partial^2 v_i + f_i \qquad (28)$$

- For Reynolds number $Re = LV/\nu \ll 100$ smooth laminar flow
- However, when $Re \gg 100$ onset of turbulence. Anomaly: energy dissipation doesn't vanish
- Highly non-linear, random, dofs strongly coupled
- Need statistical description- random force f_i

Kolmogorov theory ($d = 3$)

- Kolmogorov: Energy injected at large scales L flows to smaller scale L_{diss}. Large eddies break down to small ones
- Inertial Range, Universality, Scale invariance
  $L \gg L_{diss}$ effects of both external forcing and viscosity small. Dissipative anomaly.

$$S_n(r) \equiv \left\langle \left((\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) \cdot \frac{\mathbf{r}}{r} \right)^n \right\rangle = C_n \langle \epsilon^{n/3} \rangle r^{n/3} \tag{29}$$

$\mathbf{r} = \mathbf{x} - \mathbf{y}$

- Scale invariance *not true* for higher moments
- One exact result $n = 3$ ($C_3 = -\frac{4}{5}$). Power spectrum for fluid velocity $E(k) \sim k^{-5/3}$
- 2d fluids are different....

2d turbulence

- Enstrophy ω^2 (and powers of it) are conserved $\int \omega^2 d^2 x$
  $\langle \epsilon \rangle = \nu \langle \omega^2 \rangle$
  Kraichnan: $d = 2$ Enstrophy cascades directly (to smaller scales), Energy obeys now an *inverse* cascade (to large scales)
- $S_3 = \frac{2}{3} \langle \epsilon \rangle r$, $E(k) \sim k^{-5/3}$
- Inverse cascade statistics is scale invariant...
- Long lived vortices

2+1 dimensional ideal hydro

- An additional relativistic conserved current $\partial_\mu J^\mu = 0$ (Carrasco, et. al 1210.6702)

$$\Omega_{\mu\nu} = \xi \epsilon_{\mu\nu\lambda} u^\lambda, \quad J^\mu = T^{-2} \Omega_{\alpha\beta} \Omega^{\alpha\beta} u^\mu \tag{30}$$

- Non-relativistic case: vorticity

$$\Omega_{\mu\nu} \to T_0 \omega_{ij} \tag{31}$$

$$\omega_{ij} = 2\partial_{[i} v_{j]}, \quad \omega = \epsilon^{ij} \omega_{ij} \tag{32}$$

- $\partial_t \omega + v^i \partial_i \omega = 0$
  Both $Z = \int d^2x \frac{\omega^2}{2}$ and $E = \int d^2x \frac{v^2}{2}$ conserved in absence of friction

Some holographic statements

- Generally, in the inertial range, dual black hole horizon is *non-expanding*. Fluctuations preserve cross-sectional area
- Generally, the horizon should have random, fractal nature Eling, Fouxon, Oz 1004.2632
- Difference between 2d and higher d turbulence: gravitational perturbations should behave differently in 4d than in higher dimensions Evidence of last two seen recently numerically in 4*d* black brane (Adams, Chesler, Liu 1307.7267)
- What can we say about enstrophy/vorticity in the gravity dual?

Geometric, gauge invariant characterization

- Using Riemann tensor identities, and $R_{\mu B}\ell^B = 0$ one can show

$$2\nabla_{[\mu}c_{\nu]}\ell^C = -R_{\mu\nu DC}\ell^C = -C_{\mu\nu DC}\ell^C \tag{33}$$

- Introduce null tetrad basis $(\ell^A, n_A, m^A, \bar{m}^A)$

$$\ell_A = (1,0), \ell^A = (0, u^\mu); \quad n_A = (0, u_\mu), n^A = (1,0) \tag{34}$$

One finds

$$\nabla_{[\mu}c_{\nu]} = \frac{1}{2}C^{(1)}_{\mu\nu\lambda r}u^\lambda = 2i\mathrm{Im}\Psi_2 m_{[\mu}\bar{m}_{\nu]} \tag{35}$$

where $\Psi_2 = C_{ABCD}\ell^A m^B \bar{m}^C n^D$.

- Non-relativistic limit

$$\omega = \frac{1}{2T_0}\mathrm{Im}\Psi_2 \tag{36}$$

- Second variable characterizing horizon is intrinsic scalar curvature (Ashtekar, et. al 2004; Penrose/Rindler)

$$\Phi_H = \frac{1}{4}\tilde{R} - i\mathrm{Im}\Psi_2 \tag{37}$$

Find that generically

$$\mathrm{Re}\Phi_H^{(1)} \sim \frac{\partial_\lambda u^\lambda}{T} \tag{38}$$

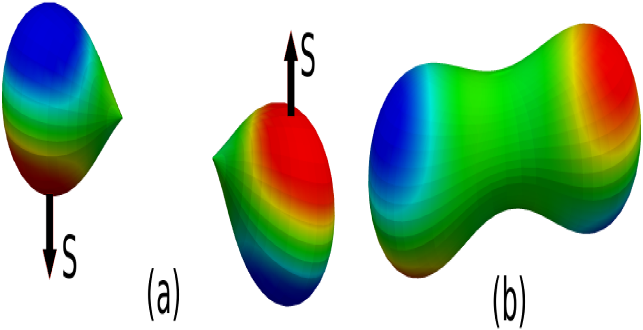
and in non-relativistic limit

$$\mathrm{Re}\Phi_H^{(1)} \sim \partial_i v^i \ . \tag{39}$$

- $\mathrm{Im}\Psi_2$ completely characterizes horizon geometry in non-relativistic case
- This variable is gauge invariant- independent of how you choose tetrad (Lorentz rotations)

Numerical GR

- Horizon vorticity and “tendicity" can be found numerically



- Taken from 1012.4869, R. Owen, et.al

Geometrical scalings

- Direct cascade
  No scale invariance

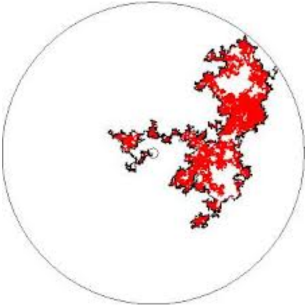
$$\langle \omega^n(\vec{r}, t)\omega^n(0, t)\rangle \sim \left[D \ln\left(\frac{L}{r}\right)\right]^{\frac{2n}{3}} \tag{40}$$

$$E(k) \sim D^{\frac{2}{3}} k^{-3} \ln^{-\frac{1}{3}}(kL) \; . \tag{41}$$

We expect Log structure in $\mathrm{Im}\Psi_2$.

Inverse cascade: Zero vorticity lines and SLE curves

- Zero vorticity lines $\rightarrow \mathrm{Im}\Psi_2 = 0$
- Kraichnan scaling $v \sim r^{1/3}$ and $\omega \sim r^{-2/3}$ implies $d_{fractal} = \frac{4}{3}$
- shown to be random SLE curves $\rightarrow$ conformal invariance (Bernard, Boffetta, Celani, Falkovich 2006)



- Universal scale and conformal structures in 2d cascades rooted in CFT fluid flows? (role of Weyl tensor here)

Random surfaces

- Bulk geometry dual to turbulent state is random
- Induces a random horizon structure. Can we gain clues about turbulence from theory of random surfaces?
- Horizon cross-section is a 2d random surface- Polyakov, et.al in 1980's

$$S_{polya} = \int d^2x \sqrt{-g}\tilde{R}\frac{1}{\Box}\tilde{R} \qquad (42)$$

$$F = \int d[g] e^{-S_{polya}} \qquad (43)$$

- Conformal gauge- $e^{2\phi}\delta_{ij} \to$ (quantum) Liouville gravity
- Anomalous scalings- operators coupled to 2d quantum gravity KPZ and DDK 1988-9

Extrinsic curvature

- Horizon has extrinsic curvature K Solodukhin 2011

$$S = \int d^2x \sqrt{-g} \left(a\,\tilde{R} \frac{1}{\Box} \tilde{R} + b\,K \frac{1}{\Box} \tilde{R} + W_{conf} \right) \tag{44}$$

- Conformal fluctuations don't affect vorticity... maybe conformally invariant part is relevant here?

Discussion/Speculation

- Hints of conformal invariance in 2d turbulence?
- Non-expanding horizon reminiscent of role of area preserving diffeos in study of Euler equation (Arnold)
- Question of finite time singularities in 3d NS equation $\rightarrow$ cosmic censorship?

Conclusion

- Interplay between geometry and fluid physics
- 2d fluid vorticity mapped into gauge invariant observable characterizing horizon geometry

Even though we have some exotic, strongly coupled CFT fluid, universality means holography is relevant for real world turbulence?!