# *All-loop S-matrix of planar $\mathcal{N}=4$ Super Yang-Mills from Yangian symmetry*

Song He

*Simon Caron-Huot & SH, arXiv: 1112.1060.*

*Crete Center for Theoretical Physics, University of Crete*

*April 24th, 2012*

## Plan of the talk

- Motivations
- Review of the S-matrix in planar $\mathcal{N} = 4$ SYM
- The S-matrix from symmetries
  - New differential equations
  - Solving the equations
- Jumpstarting amplitudes
  - Two-loop MHV
  - Two-loop NMHV
  - Three-loop MHV
  - Two-dimensional kinematics
- Outline of a derivation
- Summary and outlook

## Motivations

- Integrability in AdS/CFT duality
  - Planar $\mathcal{N} = 4$ SYM and IIB superstring theory on $AdS_5 \times S^5$ are integrable.
  - From dimensions of local operators to other important observables, such as correlation functions, Wilson loops and scattering amplitudes.
  - Integrability as a hidden, infinite-dimensional symmetry: the $\mathfrak{psu}(2,2|4)$ Yangian.
- S-matrix program: $\mathcal{N} = 4$ SYM as the simplest QFT
  - Remarkable structures of amplitudes in gauge theories and gravity, which are completely obscured in textbook formulation of QFT.
  - Planar $\mathcal{N} = 4$ SYM has the nicest S-matrix, and can be viewed as our new harmonic oscillator. It serves as a toy model for general gauge theories and gravity.
  - Towards a dual formulation of QFT from S-matrix program, which manifests the structures of amplitudes: *symmetries constrains everything?*

## Review of the S-matrix in planar $\mathcal{N}=4$ SYM

- All the on-shell states in $\mathcal{N}=4$ SYM can be combined into an on-shell superfield,

$$\Phi := G^{+} + \eta^{A}\Gamma_{A} + \frac{1}{2!}\eta^{A}\eta^{B}S_{AB} + \frac{1}{3!}\varepsilon_{ABCD}\eta^{A}\eta^{B}\eta^{C}\bar{\Gamma}^{D} + \frac{1}{4!}\varepsilon_{ABCD}\eta^{A}\eta^{B}\eta^{C}\eta^{D}G^{-},$$

which depends on the Grassmann variable $\eta^{A}$, and a null momenta $p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\bar{\lambda}_{\dot{\alpha}}$.

- All color-ordered amplitudes are then packaged into a superamplitude $\mathcal{A}(\{\lambda_i, \bar{\lambda}_i, \eta_i\})$, which has an expansion in terms of Grassmann degrees $4k+8$,

$$\mathcal{A}_n = \mathcal{A}_{n,\mathrm{MHV}} + \mathcal{A}_{n,\mathrm{NMHV}} + \cdots + \mathcal{A}_{n,\overline{\mathrm{MHV}}} = \frac{\delta^4(\sum_i \lambda_i \bar{\lambda}_i)\delta^{0|8}(\sum_i \lambda_i \eta_i)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\sum_{k=0}^{n-3} A_{n,k},$$

where $A_{n,k}$ denotes the N$^k$MHV amplitude, with MHV tree, $\mathcal{A}^{\mathrm{tree}}_{n,\mathrm{MHV}}$, stripped off.

- $\mathcal{N}=4$ SYM is a *superconformal* field theory, which should be reflected in the structure of scattering amplitudes. The tree-level S-matrix is invariant under this $\mathfrak{psu}(2,2|4)$ symmetry: $\{\mathfrak{q}^{\alpha}_{A}, \bar{\mathfrak{q}}^{A}_{\dot{\alpha}}, \mathfrak{p}_{\alpha\dot{\alpha}}, \mathfrak{m}_{\alpha\beta}, \bar{\mathfrak{m}}_{\dot{\alpha}\dot{\beta}}, \mathfrak{s}^{A}_{\alpha}, \bar{\mathfrak{s}}^{\dot{\alpha}}_{A}, \mathfrak{k}_{\alpha\dot{\alpha}}, \mathfrak{d}, \mathfrak{r}^{A}_{B}\}$. At loop level, the superconformal symmetry of the S-matrix is broken by infrared divergences.

# Review of the S-matrix in planar $\mathcal{N}=4$ SYM: dual symmetries

- A *dual* conformal symmetry has been observed at both weak [Drummond Henn Smirnov Sokatchev 2006] and strong couplings [Alday Maldacena 2007]. The symmetry has been generalized to a dual super-conformal symmetry [Drummond Henn Korchemsky Sokatchev 2008] of the dual chiral superspace,

$$x_i^{\alpha\dot\alpha} - x_{i-1}^{\alpha\dot\alpha} = \lambda_i^\alpha \bar\lambda_i^{\dot\alpha}, \qquad \theta_i^{\alpha A} - \theta_{i-1}^{\alpha A} = \lambda_i^\alpha \eta_i^A.$$

  The tree-level S-matrix is invariant under the dual $\mathfrak{psu}(2,2|4)$ symmetry.

- An all-loop, exponentiated ansatz for MHV amplitude in $4-2\epsilon$ dimensions has been proposed, which encodes infrared and collinear behavior [Anastasiou Bern Dixon Kosower 2003] [Bern Dixon Smirnov 2005],

$$A_n^{\mathrm{BDS}} = 1 + \sum_{\ell=1}^{\infty} g^{2\ell} A_n^{(\ell)}(\epsilon) := \exp\left[\sum_{\ell=1}^{\infty} g^{2\ell}\left(\Gamma_{\text{cusp}}^{(\ell)}(\epsilon) A_{n,0}^{(1)}(\ell\epsilon) + C^{(\ell)} + E_n^{(\ell)}(\epsilon)\right)\right].$$

- MHV loop amplitudes satisfy an anomalous Ward identity for the dual conformal symmetry [Drummond Henn Korchemsky Sokatchev 2007]. For $n=4,5$, the only solution is given by the BDS ansatz, since there is no cross-ratios. A finite remainder function of $3(n-5)$ cross-ratios is allowed for $n$-point MHV amplitude, e.g. $u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$ etc. for $n=6$.

# Review of the S-matrix in planar $\mathcal{N}=4$ SYM: Wilson loops

- There is strong evidence for a duality between MHV amplitude and a null polygonal Wilson loop in dual spacetime [Drummond Korchemsky Sokatchev 2007] [Brandhuber Heslop Travaglini 2007] [Bern Dixon Kosower Roiban Spradlin Vergu Volovich 2008]. On the string side, (fermionic) T-duality maps the original superconformal symmetry of the amplitude to the dual symmetry of the Wilson loop [Berkovits Maldacena 2008] [Beisert Ricci Tseytlin Wolf 2008], and their closure is the Yangian symmetry, $y[\mathfrak{psu}(2,2|4)]$ [Drummond Henn Plefka 2009].

- A generalized duality between the superamplitude and a supersymmetric Wilson loop has been derived at the integrand level [Mason Skinner 2010][Caron-Huot 2010], although a rigorous UV regularization for the super-loop has not been carried out [Belitzky Korchemsky Sokatchev 2011],

$$A_n(\lambda_i, \bar{\lambda}_i, \eta_i) = W_n(x_i, \theta_i)(1+\mathcal{O}(\epsilon)), \quad W_n = \frac{1}{N_c}\langle \mathrm{Tr}\mathcal{P}e^{-\oint \mathbf{A}(x_i,\theta_i)}\rangle.$$

- The super Wilson loop in chiral formalism obscures one chiral half of superconformal symmetries. As a natural generalization, Wilson loops in non-chiral $\mathcal{N}=4$ superspace generally manifest the full symmetry [Caron-Huot 2011] [Beisert Vergu 2012] [Beisert SH Schwab Vergu 2012].
- One can obtain amplitudes by setting $\bar{\theta}=0$, but there is no obvious way to define non-chiral amplitudes dual to non-chiral Wilson loops. They contain additional terms, which can play a role for compensating symmetry anomalies of amplitudes.

## Review of the S-matrix in planar $\mathcal{N}=4$ SYM: momentum twistors

- It is convenient to introduce unconstrained *momentum-twistor* variables [Hodges 2009],

$$\mathcal{Z}_i = (Z_i^a, \chi_i^A) := (\lambda_i^\alpha, x_i^{\alpha\dot\alpha}\lambda_{i\alpha}, \theta_i^{\alpha A}\lambda_{i\alpha}),$$

which are twistors of the dual (super)space. Then one can construct invariants,

$$\text{four-bracket}: \quad \langle ijkl\rangle := \varepsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d, \quad \text{e.g.} \quad u_1 = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle}.$$

$$\text{R-invariant}: \quad [i\,j\,k\,l\,m] := \frac{\delta^{0|4}(\chi_i^A\langle jklm\rangle + \text{cyclic})}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}.$$

- Using momentum twistors, which form fundamental representation of the dual $\mathfrak{psu}(2,2|4)$, all the generators become first-order differential operators,

$$Q_A^a = (\mathfrak{Q}_A^\alpha, \bar{\mathfrak{S}}_A^{\dot\alpha}) := \sum_{i=1}^n Z_i^a \frac{\partial}{\partial \chi_i^A}, \qquad \bar{Q}_a^A = (\mathfrak{S}_\alpha^A, \bar{\mathfrak{Q}}_{\dot\alpha}^A = \bar{s}_{\dot\alpha}^A) := \sum_{i=1}^n \chi_i^A \frac{\partial}{\partial Z_i^a},$$

$$K_b^a = (\mathfrak{P}_{\alpha\dot\alpha}, \mathfrak{K}_{\alpha\dot\alpha}, \mathfrak{M}_{\alpha\beta}, \bar{\mathfrak{M}}_{\dot\alpha\dot\beta}, \mathfrak{D}) := \sum_{i=1}^n Z_i^a \frac{\partial}{\partial Z_i^b}, \qquad R_B^A = \mathfrak{R}_B^A := \sum_{i=1}^n \chi_i^A \frac{\partial}{\partial \chi_i^B}.$$

## Review of the S-matrix in planar $\mathcal{N}=4$ SYM: current status

- All tree amplitudes are known by BCFW recursions [Britto Cachazo Feng 2004], e.g. NMHV tree, $A^{\text{tree}}_{n,1} = \sum_{1<i<j<n}[1\,i\,i{+}1\,j\,j{+}1]$, which are built from leading singularities, or (generally-shifted) R-invariants. All leading singularities are Yangian invariant, and correspond to contour integrals on the Grassmannian $G(k,n)$ [Arkani-Hamed Cachazo Cheung Kaplan 2009].

- The Yangian-invariant planar integrand of all-loop amplitudes/Wilson loops is known recursively [Arkani-Hamed Bourjaily Cachazo Caron-Huot Trnka 2010], but it is difficult to perform integrals.

- The $n$-point, N$^k$MHV, $\ell$-loop amplitude is of the form "$G(k,n)$ leading-singularities" $\times$ "pure, transcendental degree $2\ell$ functions of $3(n{-}5)$ cross-ratios". It is convenient to use "symbol" for transcendental functions as iterated integrals [Goncharov Spradlin Vergu Volovich 2010],

$$F(x) = \int_{x_1<\cdots<x_m<x} d\log X_1(x_1)\ldots d\log X_m(x_m) \Rightarrow S[F] = X_1 \otimes \ldots \otimes X_m .$$

- All one-loop amplitudes are known using leading singularity method (or generalized-unitarity), but higher-loop integral basis are lacking. Recent advances have reached two-loop MHV [Del Duca Duhr Smirnov 2010][Goncharov Spradlin Vergu Volovich 2010], NMHV and three-loop MHV [Kosower Roiban Vergu 2011][Dixon Drummond Henn 2011][Caron-Huot SH 2011]. It is promising to go to all loops in the near future!

# The S-matrix from symmetries: new differential equations

- We define *BDS-subtracted* S-matrix: $A_{n,k} = A_n^{\rm BDS} \times R_{n,k}$, which is finite, depends on conformal cross-ratios and R-invariants, and has simple collinear limits: the $k$-preserving limit, $R_{n,k} \to R_{n-1,k}$, and the $k$-decreasing one, $\frac{\int d^4\chi_n R_{n,k}}{\int d^4\chi_n [n{-}2\, n{-}1\, n\, 1\, 2]} \to R_{n-1,k-1}$. By construction, $R_{4,0} = R_{5,0} = R_{5,1}/R_{5,1}^{\rm tree} = 1$.

- The BDS-subtracted S-matrix is invariant under $Q_A^a, R_B^A, K_b^a$, but not for (naive) $\bar{Q}_a^A$. We propose an all-loop equation in terms of collinear integral (see also [Bullimore Skinner 2011]),

$$\bar{Q}_a^A R_{n,k} = \Gamma_{\rm cusp}\, {\rm res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(d^{2|3} \mathcal{Z}_{n+1}\right)_a^A \left[R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\rm tree}\right] + \text{cyclic},$$

where the cusp anomalous dimension is known $\Gamma_{\rm cusp} = g^2 - \frac{\pi^2}{3} g^4 + \frac{11\pi^4}{45} g^6 + \ldots$.

- For $\mathcal{Z}_{n+1}$, we integrate over $0 \leq \tau < \infty$, and extract the coefficient of $d\epsilon/\epsilon$ as $\epsilon \to 0$,

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon(\mathcal{Z}_{n-1} - \tau C \mathcal{Z}_1) + \mathcal{O}(\epsilon^2), \quad C := \frac{\langle n{-}1 n 2 3\rangle}{\langle n 1 2 3\rangle},$$

$${\rm res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} (d^{2|3} \mathcal{Z}_{n+1})_a^A = C\, \bar{n}_a \oint_{\epsilon=0} \epsilon\, d\epsilon \int_0^\infty d\tau (d^{0|3} \chi_{n+1})^A, \quad (i{-}1 i i{+}1) := \bar{i}.$$

# The S-matrix from symmetries: new differential equations

- Using the discrete parity symmetry, we derive an equivalent equation for level-one generator, $Q_A^{(1)a} = (s_A^\alpha, \ldots) := \frac{1}{2}\sum_{i,j} \mathrm{sgn}(j-i)\left(Z_i^a \frac{\partial}{\partial Z_i^b} Z_j^b \frac{\partial}{\partial \chi_j^A} - Z_i^a \frac{\partial}{\partial \chi_i^B} \chi_j^B \frac{\partial}{\partial \chi_j^A}\right)$,

$$Q_A^{(1)a} R_{n,k} = \Gamma_{\text{cusp}} Z_n^a \lim_{\epsilon \to 0} \int_0^\infty \frac{d\tau}{\tau} (d\eta_{n+1})_A \left( R_{n+1,k} - \sum_{i,j} C_{i,j} \frac{\partial R_{n,k}}{\partial \chi_j} \right) + \text{cyclic.}$$

- The differential equations are finite, regulator independent, and manifest the transcendentality of loop amplitudes. On the RHS, the measures of integrating out a particle carry correct quantum numbers, and $1d$ integrals reflect that naive generators are violated since they cause asymptotic states to radiate collinearly.

- Given RHS of both equations as linear operators acting on S-matrix, they can be interpreted as *quantum corrections* to the naive generators [Bargheer Beisert Galleas Loebbert McLoughlin 2009] [Sever Vieira 2009], in which sense the BDS-subtracted S-matrix is Yangian invariant!

- We claim the equations to be valid for any value of the coupling (the explicit dependence is only through $\Gamma_{\text{cusp}}$), and they determine the all-loop S-matrix.

## The S-matrix from symmetries: solving the equations

- The RHS of $\bar{Q}$ equation can be evaluated at the $\tau$-integrand level, $X = Z_n \wedge Z_{n+1}$,

$$\int d^{2|3}\mathcal{Z}_{n+1}[i\,j\,k\,n\,n{+}1]f(\tau,\epsilon) = \bar{Q}\log\frac{\langle \bar{n}j\rangle}{\langle \bar{n}i\rangle}\int_0^\infty d\log\frac{\langle Xij\rangle}{\langle Xjk\rangle}f(\tau,\epsilon\to 0) + (j \leftrightarrow k),$$

  and other R-invariants give vanishing result. For the $\bar{Q}$ of all one-loop N$^k$MHV amplitudes, the RHS comes from tree amplitudes, where it is easy to perform the $\tau$-integral, and the result agrees with [Beisert Henn McLoughlin Plefka 2010].

- For MHV amplitude, since $R_{n,0}$ is independent of Grassmann variables, $\bar{Q}$ equation gives all derivatives, $\frac{\partial}{\partial \chi_i^1}\bar{Q}_a^1 = \frac{\partial}{\partial Z_i^a}$, and uniquely determine MHV amplitudes up to a constant, to be fixed by a collinear limit. The total derivative of MHV remainder is $dR_{n,0} = \sum_{i,j} F_{i,j}\, d\log\langle \bar{i} j\rangle$, which proves the conjecture of [Caron-Huot 2011].

- Similarly NMHV is uniquely determined by $\bar{Q}$ equation up to a linear combination of R-invariants, which is fixed by collinear limits. Beyond NMHV level, we also need to use $Q^{(1)}$ equation. All invariant under naive $Q$, $\bar{Q}$ and $Q^{(1)}$ are given by leading singularities [Korchemsky Sokatchev 2010][Drummond Ferro 2010], thus, up to such invariants, all-loop N$^k$MHV amplitudes are determined by both equations!

- The $\bar{Q}$ of two-loop MHV hexagon is given by the collinear integral of $R_{7,1}^{\text{1-loop}}$,

$$\bar{Q}R_{6,0}^{\text{2-loop}} = (I_1 + I_1^\epsilon)\bar{Q}\log\frac{\langle 5613\rangle}{\langle 5612\rangle} + (I_2 + I_2^\epsilon)\bar{Q}\log\frac{\langle 5614\rangle}{\langle 5612\rangle} + \text{cyclic},$$

where it is of paramount importance to us that upon $\tau$-integral $I_{1,2}$ and $I_{2,2}$ vanish,

$$I_1^\epsilon = \log\epsilon^2 \times \int_0^\infty d\left(\log\frac{u_3(\tau+1)}{\tau+u_3}\log(\frac{\tau}{\tau+u_3}) + \log(\tau+1)\log\frac{\tau+u_3}{\tau+1}\right) = 0,$$
$$I_2^\epsilon = \log\epsilon^2 \times \int_0^\infty d\left(\log\frac{\tau+u_3}{\tau}\log\frac{u_3}{\tau+u_3}\right) = 0.$$

It is straightforward to obtain the finite integrals, in terms of 6D hexagon integral,

$$I_1 = \left(\frac{1}{3}\log^2 u_3 + \log u_1 \log u_2 + \sum_{i=1}^{3}\text{Li}_2(1-u_i)\right)\log u_3 - 2\text{Li}_3(1-\frac{1}{u_3}),$$
$$I_2 = -\frac{1}{2}I_6^{6D} + \sum_{i=1}^{3}(-)^{\delta_{3i}}\text{Li}_3(1-\frac{1}{u_i}) + \frac{1}{2}\log\frac{u_2u_3}{u_1}\sum_{i=1}^{3}\text{Li}_2(1-\frac{1}{u_i}) + \frac{1}{12}\log^3\frac{u_2u_3}{u_1}.$$

- Therefore, we obtain the total differential of $R_{6,0}^{\text{2-loop}}$ in a very compact form,

$$dR_{6,0}^{\text{2-loop}} = I_6^{6D}\, d\log\frac{x^+}{x^-} + \left(I_1\, d\log\frac{1-u_3}{u_3} + \text{two cyclic images}\right),$$

which can be integrated and agrees precisely with [Del Duca Duhr Smirnov 2010][Goncharov Spradlin Vergu Volovich 2010],

$$R_{6,0}^{\text{2-loop}} = 4\sum_{i=1}^{3}\left(L_4^+(u_i) - \frac{1}{2}\mathrm{Li}_4(1-\frac{1}{u_i})\right) - \frac{1}{2}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-\frac{1}{u_i})\right)^2 + \frac{1}{6}J^4 + \frac{\pi^2}{3}J^2 + \frac{\pi^4}{18}.$$

- There is no qualitative difference between $n > 6$ cases and the hexagon. The $\log\epsilon^2$ terms integrate to zero, leaving finite, conformal integrals, which can be easily evaluated at the level of symbol. The result agrees with [Caron-Huot 2011] up to $n = 10$.

- Furthermore, we can choose an integral path connecting a collinear $(n-1)$-gon to the original $n$-gon, and obtain an integral representation for two-loop $n$-point MHV. We hope to compare [Caron-Huot SH unpublished 2011] with numerical results in [Anastasiou Brandhuber Heslop Khoze Spence Travaglini 2009].

# Jumpstarting amplitudes: two-loop NMHV

- NMHV hexagon is given by collinear integral of $\text{N}^2$MHV heptagon. From its leading singularities, we get $7 \times 6$ prefactors, and conformal symmetry removes one,

$$dR_{6,1} = \sum_{i=1}^{41} (\text{R-invariant})_i \times F_i \times d\log(\text{cross-ratios})_i,$$

which holds to all loops! We compute $F_i$ at two-loop and put it in the following form,

$$R_{6,1}^{\text{2-loop}} = [(1){+}(4)]V_3{+}[(2){+}(5)]V_1{+}[(3){+}(6)]V_2{+}[(1){-}(4)]\tilde{V}_3{+}[(5){-}(2)]\tilde{V}_1{+}[(3){-}(6)]\tilde{V}_2$$

where $V$'s and $\tilde{V}$'s are degree-$4$ functions, with differentials as follows,

$$dV_3 = -\frac{1}{2} I_6^{6D} d\log\frac{y_2}{y_3} + (dV_3)_1 d\log\frac{u_1}{(1-u_2)(1-u_3)} + (dV_3)_2 d\log\frac{1-u_1}{u_2 u_3}$$
$$+\left((dV_3)_3 d\log\frac{u_2}{1-u_2} + (u_2 \leftrightarrow u_3)\right),$$
$$d\tilde{V}_3 = \frac{1}{2} I_6^{6D} d\log\frac{u_2(1-u_3)}{(1-u_2)u_3} + (d\tilde{V}_3)_1 d\log y_1 + (d\tilde{V}_3)_2 d\log y_2 y_3 + (d\tilde{V}_3)_3 d\log\frac{y_2}{y_3}.$$

We find the *function* agrees with the results in [Kosower Roiban Vergu 2011] and [Dixon Drummond Henn 2011].

# Jumpstarting amplitudes: three-loop MHV

- Similarly we can obtain higher-point NMHV amplitudes at two loops. For heptagon, there are 288 independent prefactors, and its symbol has been obtained, but the complexity increases rapidly with $n$ [Caron-Huot SH unpublished 2011].

- Based on physical considerations and assumptions on the symbol, an ansatz for the symbol of three-loop MHV hexagon was proposed [Dixon Drummond Henn 2011]. From NMHV hexagon we confirm the assumptions, and fix the two undetermined parameters,

$$S[R_{6,0}^{\text{3-loop}}] = \left( S[X] - \frac{3}{8} S[f_1] + \frac{7}{32} S[f_2] \right) (u_1, u_2, u_3),$$

- It is possible that by fixing two-loop N$^2$MHV (e.g. N$^2$MHV heptagon is given by the parity conjugate of NMHV, and the octagon is reachable), one could obtain the symbol of three-loop NMHV and even four-loop MHV using $\bar{Q}$ equations.

- Furthermore, one can make all-loop predictions, e.g. determine the final entry of NMHV symbol, which in turn gives the next-to-final entry of MHV symbol. Together with $Q^{(1)}$ equation, we can certainly go on in the higher $n, k, \ell$ direction.

- It is useful to consider $(2n)$ external momenta embedded in a two dimensional sub-space, which reduces the superconformal group $SU(2,2|4)$ to $SL(2|2)\times SL(2|2)$,

$$\mathcal{Z}_{2i-1}=(\lambda_i^{1+},0,\lambda_i^{2+},0,\chi_i^{1+},0,\chi_i^{2+},0), \qquad \mathcal{Z}_{2i}=(0,\lambda_i^{1-},0,\lambda_i^{2-},0,\chi_i^{1-},0,\chi_i^{2-}).$$

  The only non-vanishing four-brackets are $\langle 2i{-}1\, 2j{-}1\, 2k\, 2l\rangle=\langle ij\rangle^+\langle kl\rangle^-$, and we can define odd and even cross-ratios $u^\pm_{i,j}:=\frac{\langle ij{+}1\rangle^\pm\langle i{+}1j\rangle^\pm}{\langle ij\rangle^\pm\langle i{+}1j{+}1\rangle^\pm}$.

- For superamplitude, R-invariants also factorize into odd and even parts, $[*\,2i{-}1\,2j{-}1\,2k\,2l]=(*\,i\,j)[*\,k\,l]$ where $(*\,i\,j):=\frac{\delta^{0|2}(\langle\langle *\,i\,j\rangle\rangle^+)}{\langle *\,i\rangle^+\langle i\,j\rangle^+\langle j\,*\rangle^+}$ and similarly for $[*\,k\,l]$, which all satisfy $(a\,b\,c)-(a\,b\,d)+(a\,c\,d)-(b\,c\,d)=0$. In this notation, the NMHV tree is $R^{\text{tree}}_{2n,1}=\frac{1}{2}\sum_{i,j}(*\,i\,j)\,([i\,j{-}1\,j]-[i{-}1\,j{-}1 j])$.

- The natural collinear limit in 2d is a triple-collinear limit, $R_{2n}\to f(g^2)R_{2n-2}$, and it is convenient to rescale the BDS-subtracted amplitude so that it has natural $k$-preserving and decreasing limits, $R_{2n,k}:=e^{(n-2)f_1(g^2)+kf_2(g^2)}\tilde{R}_{2n,k}$, where

$$f_1(g^2)=-\Gamma^2_{\text{cusp}}\frac{\pi^4}{9}+\mathcal{O}(\Gamma^3_{\text{cusp}}),\quad f_2(g^2)=-\Gamma_{\text{cusp}}\frac{\pi^2}{3}+\Gamma^2_{\text{cusp}}\frac{7\pi^4}{30}+\mathcal{O}(\Gamma^3_{\text{cusp}}).$$

# Jumpstarting amplitudes: two-dimensional kinematics

- In the even sector, the $\bar{Q}$ equation in 2d is ($\lambda^{\pm}_{n+1}=\lambda^{\pm}_{n}+\epsilon\lambda^{\pm}_{1}, \chi^{\pm}_{n+1}=\chi^{\pm}_{n}+\epsilon\chi^{\pm}_{1}$),

$$\bar{Q}^{A}_{a}\tilde{R}_{2n,k}=\Gamma_{\text{cusp}}\int d^{1|2}\lambda^{+}_{n+1}\int d^{0|1}\lambda^{-}_{n+1}(\tilde{R}_{2n+2,k+1}-R^{\text{tree}}\tilde{R}_{2n,k})+\text{cyclic}.$$

- One can easily write down N$^2$MHV tree, including the degenerate terms,

$$R^{\text{tree}}_{2n,2}=\frac{1}{2}\sum_{i<j<k<l<i}(*\,i\,j)(*\,k\,l)\,([i\,j{-}1\,j]-[i{-}1\,j{-}1j])\,([k\,l{-}1\,l]-[k{-}1\,l{-}1l])+\ldots,$$

from which we can get one-loop NMHV, and the two-loop MHV amplitude,

$$\tilde{R}^{\text{2-loop}}_{2n,0}=-\sum_{i<j<k<l<i}\log\langle ik\rangle\log\langle jl\rangle\log u^{-}_{i-1,k-1}\log u^{-}_{j-1,l-1}-2\sum_{i<j<k<i}\log\langle ij\rangle\log\langle jk\rangle$$
$$\times\,(\log u^{-}_{j-1,k-1}\log u^{-}_{k-1,i,i-1,j}+\text{cyclic})-\sum_{i<j<i}\log^{2}\langle ij\rangle\log u^{-}_{i-1,j-1}\log(1-u^{-}_{i-1,j-1}).$$

Nothing prevents us from getting $n$-point two-loop NMHV and three-loop MHV. Main complexity of loop amplitudes in 2d already hidden inside tree amplitudes?

# Outline of a derivation

- A heuristic derivation expresses the RHS of $\bar{Q}$ equation in terms of a fermion excitation inserted on each edge of the Wilson loop (see [Bullimore Skinner 2011] from twistor viewpoint),

$$\bar{Q}^A_{\dot{\alpha}}\langle W_n\rangle \propto g^2 \oint dx_{\dot{\alpha}\alpha}\langle(\psi^A + F\theta^A + \ldots)^\alpha W_n\rangle,$$

  which was calculated in explicit examples by Feynman diagrams [Caron-Huot 2011].

- The key new ingredient: there is only one fermionic excitation of null edges with given quantum numbers. The Operator Product Expansion [Alday Gaiotto Maldacena Sever Vieira 2010] allows us to extract the excited $n$-gon Wilson loop from an $(n+1)$-gon in collinear limit,

$$\frac{1}{A_n^{\text{BDS}}}\bar{Q}\langle W_{n,k}\rangle = \frac{g^2}{F(g^2)}\,\text{res}_{\epsilon=0}\int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_{n+1}R_{n+1,k+1}(\tau,\epsilon) + \text{cyclic}.$$

  Given that BDS ansatz is one-loop exact, we obtain the $\bar{Q}$ of BDS,

$$\langle W_{n,k}\rangle\bar{Q}\frac{1}{A_n^{\text{BDS}}} = -\Gamma_{\text{cusp}}R_{n,k}\,\text{res}_{\epsilon=0}\int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_{n+1}R^{\text{tree}}_{n+1,1}(\tau,\epsilon) + \text{cyclic}.$$

- Both $\tau$-integrals diverge, and the combination is finite provided $g^2/F(g^2) = \Gamma_{\text{cusp}}$. A crucial test of the above derivation is to check the prefactor is indeed $\Gamma_{\text{cusp}}$.

# Outline of a derivation

- The first non-trivial check of the prefactor $\Gamma_{\rm cusp}$ is the two-loop NMHV hexagon,

$$R_{6,1}^{\rm tree}+\Gamma_{\rm cusp}R_{6,1}^{\text{1-loop}}+\Gamma_{\rm cusp}^2R_{6,1}^{\text{2-loop}}+\ldots=R_{6,1}^{\prime\,\rm tree}+g^2R_{6,1}^{\prime\,\text{1-loop}}+g^4(R_{6,1}^{\prime\,\text{2-loop}}-\frac{\pi^2}{3}R_{6,1}^{\prime\,\text{1-loop}})+\ldots.$$

  The fermion excitation can be labeled by a momentum, $p$, and we can read off

$$f(\epsilon,p)=\int_0^\infty d\tau\,\tau^{i\frac{p}{2}}\,d^{0|3}\chi_6\,R_{6,1}(\epsilon,\tau).$$

  For twist-one fermion insertion, OPE predicts $\lim_{\epsilon\to 0}f(\epsilon,p)=\log\epsilon\times\gamma(p)+C(p)$, where $\gamma(p)$ is the dispersion relation known for any couplings by integrability [Basso 2010].
- From $R_{6,1}^{\text{2-loop}}$, we obtain $\gamma(p)$ to order $\Gamma_{\rm cusp}^2$, and we find agreement including $\pi^2$'s!

$$\gamma(p)=\Gamma_{\rm cusp}\,(\psi_+-\psi(1))-\frac{\Gamma_{\rm cusp}^2}{8}\left(\psi_+''+4\psi_-'(\psi_--\frac{1}{p})+6\zeta(3)\right).$$

  The cancelation of $\log\epsilon$ in total-$\tau$ integral is guaranteed: the zero-momentum fermion is the Goldstone fermion for the symmetry breaking, thus $\gamma(0)=0$.

## Summary and outlook

- The all-loop S-matrix in planar $\mathcal{N} = 4$ SYM is invariant under a suitably deformed Yangian symmetry at the quantum level, and is fully determined by it.
- We derive new, elegant equations based on the quantum-corrected symmetry, and test them extensively against e.g. results of loop amplitudes and OPE.
- We expect the equations to provide a powerful engine for computation of multi-loop amplitudes, and insights into the integrability of the theory.
- Open questions
  - Getting the actual functions, from symbol, or directly from the integral?
  - Corrected Yangian invariants from non-chiral Wilson loops [Beisert SH Schwab Vergu 2012]?
  - Construct quantum-corrected transfer matrices for the Yangian symmetry? Strong coupling tests of the equations? Relations to TBA, Y-system?
  - At least in 2d kinematics, can we imagine to compute certain amplitudes to all loops? Even as functions of the coupling constant?