Extracting Real-Time Quantities from Euclidean Field Theory

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Motivation

- by far, most hadrons in QCD are resonances rather than stable
- one would like to have an understanding of their wide range of widths
- a resonance is not an eigenstate of the Hamiltonian, but it is an enhancement of a scattering process accompanied by a large phase shift
- it is not *a priori* clear how such an effect is encoded in the Euclideanized version of the theory, where importance sampling methods can be applied
- another example of ‘real-time’ quantity: how fast does a medium like the quark-gluon plasma relax to equilibrium? ($\rightarrow$ transport coefficients).

Outline

- Minkowski and Euclidean correlation functions in Quantum Field Theory; the spectral function
- an important example: the hadronic vacuum polarization
- how to extract a spectral function from Euclidean observables **without** analytic continuation
- generalization to finite-temperature field theory
- diffusion of a heavy-quark in the quark-gluon plasma.

Euclidean Field Theory and the Spectral Representation

Spectral function:

$$\rho(\omega, \boldsymbol{k}) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{y}\; e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{y}} \langle 0 | [\mathcal{O}(t, \boldsymbol{y}), \mathcal{O}^*(0)] | 0 \rangle .$$

Euclidean correlator:

$$C_E(t, \boldsymbol{k}) \equiv \int \mathrm{d}^3\boldsymbol{y}\; e^{-i\boldsymbol{k}\cdot\boldsymbol{y}} \langle \mathcal{O}(t, \boldsymbol{y}) \mathcal{O}^*(0) \rangle = \int_0^{\infty} d\omega\; e^{-\omega t} \rho(\omega, \boldsymbol{k}) .$$

$$\rho(\omega, \boldsymbol{k}) = \mathrm{sign}(\omega) \sum_n \delta(\omega^2 - E_n^2(\boldsymbol{k})) |\langle \mathrm{vac} | \widehat{\mathcal{O}} | n, \boldsymbol{k} \rangle|^2$$

- $\rho(-\omega, \boldsymbol{k}) = -\rho(\omega, \boldsymbol{k}), \qquad \mathrm{sign}(\omega)\rho(\omega, \boldsymbol{k}) \geq 0$
- the Euclidean correlation function is the Laplace transform of the spectral density ρ

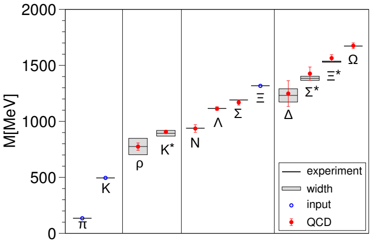
An important aspect of Euclidean Field Theory is to reconstruct ρ from the Euclidean correlation functions.

Numerical Euclidean Field Theory

- in many cases the only quantitative first-principles method available is Monte-Carlo simulations of the Euclidean field theory
- for this purpose the quantum field theory is discretized on a lattice and put in a finite volume, usually with periodic boundary conditions
- one then disposes of a finite number of data points for the correlator, with a finite statistical uncertainty
- the 'reconstruction' of the spectral density is then a numerically ill-posed problem (inverse Laplace transform)
- but, at large time separations t, the lowest energy-eigenstates dominate $\Rightarrow$ their energies (and matrix elements) can be extracted reliably.

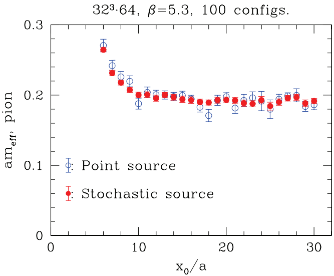
$$C_E(t,\boldsymbol{k}) = \int_0^\infty d\omega\; e^{-\omega t} \rho(\omega,\boldsymbol{k}) \stackrel{t\to\infty}{\sim} e^{-E_0(\boldsymbol{k})t}$$

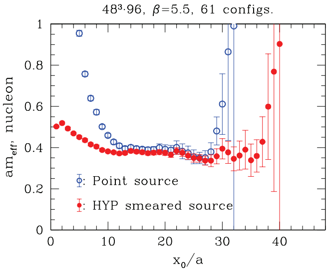
Examples of spectroscopy calculations in lattice QCD



BMW collaboration, Science 322 (2008) 1224

$$am_{\rm eff}(t) \equiv \log C(t)/C(t+a)$$

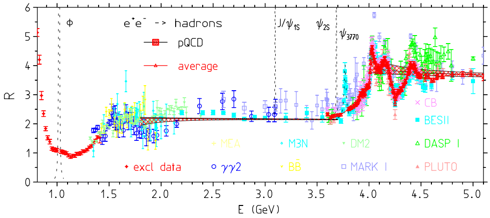




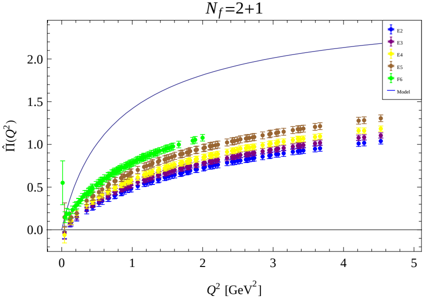
H. Wittig et al., PoS LAT2009:095,2009

Spectral function of the e.m. current vs. Euclidean correlator

[from Jegerlehner, Nyffeler 0902.3360]



[Jäger, Bernecker, Wittig, HM]



- two-point function of the electromagnetic current in QCD:

$$j_\mu(x) = \tfrac{2}{3}\bar{u}(x)\gamma_\mu u(x) - \tfrac{1}{3}\bar{d}(x)\gamma_\mu d(x) - \tfrac{2}{3}\bar{s}(x)\gamma_\mu s(x) + \ldots$$

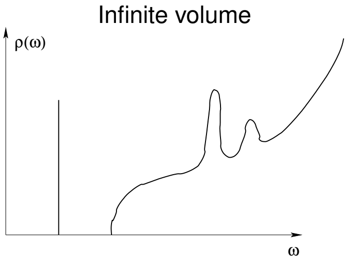
- current conservation $\Rightarrow \int d^4x \, \langle j_\mu(x) j_\nu(0)\rangle \, e^{iq\cdot x} = \Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\, \Pi(q^2)$.

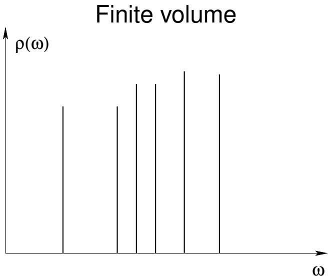
- spectral representation of vacuum polarisation: $\boxed{\Pi(q^2) - \Pi(0) = q^2 \int_0^\infty ds \frac{\rho(s)}{s(s+q^2)}}$

- via the Optical Theorem, the spectral density is accessible to experiments:

$$\pi\rho(s) = \frac{s}{4\pi\alpha(s)}\sigma_{\rm tot}(e^+e^- \to \text{everything}) = \frac{\alpha(s)}{3\pi}R(s), \quad R(s) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

Where are the real-time effects hidden?





- how can one extract
  - a scattering amplitude
  - the width of a resonance

  from Euclidean correlation functions?

- what are the finite-volume effects on the spectral density, and how does it become a continuous function when $L \to \infty$?

It turns out that these questions are related.

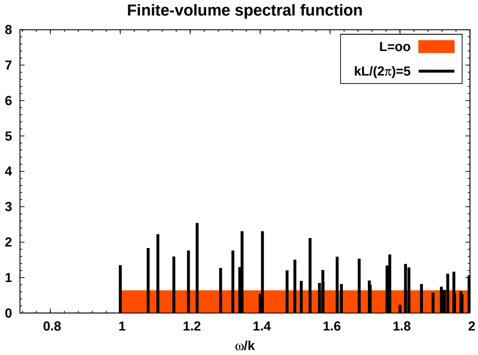
Illustration in Free Field Theory

- correlation function $C_E(t,\boldsymbol{k}) \equiv \int \mathrm{d}\boldsymbol{x}\, e^{i\boldsymbol{kx}} \langle \phi^2(t,\boldsymbol{x})\, \phi^2(0)\rangle$ (set $m=0$):

$$C(t,\boldsymbol{k}) = \frac{e^{-|\boldsymbol{k}|t}}{(4\pi)^2 t}\,, \qquad \rho(\omega,\boldsymbol{k}) = \frac{1}{(4\pi)^2}\,\theta(\omega - |\boldsymbol{k}|).$$

- in a finite periodic box:

$$C(t,\boldsymbol{k}) = \frac{1}{L^3}\sum_{\boldsymbol{p}} \frac{e^{-|\boldsymbol{p}|t}}{2|\boldsymbol{p}|}\,\frac{e^{-|\boldsymbol{k}-\boldsymbol{p}|t}}{2|\boldsymbol{k}-\boldsymbol{p}|}\,, \quad \rho(\omega,\boldsymbol{k}) = \frac{1}{L^3}\sum_{\boldsymbol{p}} \frac{\delta(\omega - |\boldsymbol{p}| - |\boldsymbol{k}-\boldsymbol{p}|)}{4|\boldsymbol{p}|\,|\boldsymbol{k}-\boldsymbol{p}|}.$$



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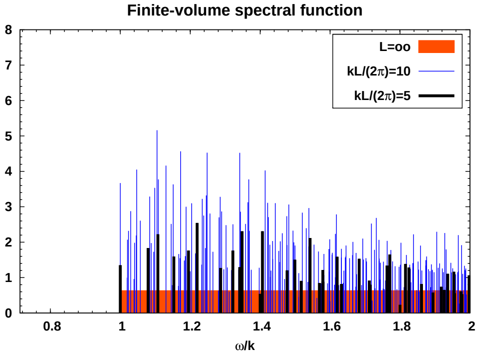

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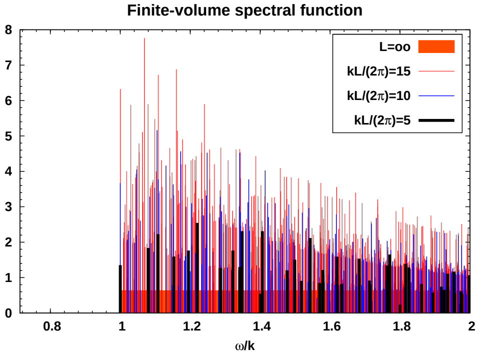
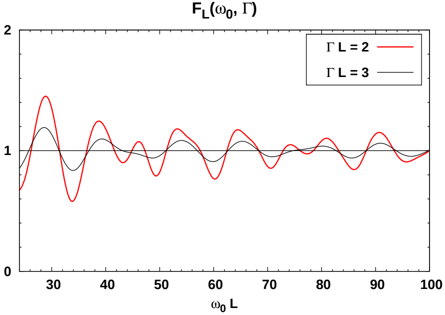


How does the spectral function behave when $L \to \infty$?



- convolution of the spectral function with a Gaussian ‘resolution function’

$$F(\omega_0, \Gamma) \equiv 4\pi^2 \int_0^\infty \mathrm{d}\omega \, \rho(\omega, \boldsymbol{k} = 0) \frac{e^{-(\omega-\omega_0)^2/2\Gamma^2}}{\sqrt{2\pi}\Gamma} \,,$$

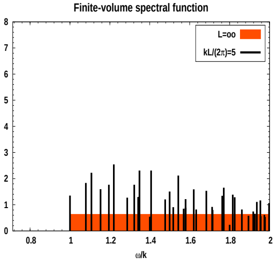
- in infinite volume, amounts to unity (for $\Gamma \ll \omega_0$)
- in finite volume, the corresponding integral amounts to

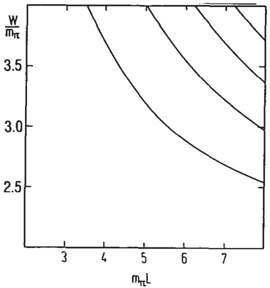
$$F_L(\omega_0, \Gamma) = \sum_{\boldsymbol{m} \in \mathbb{Z}^3} \frac{2 \sin \frac{\omega_0 |\boldsymbol{m}| L}{2}}{\omega_0 L |\boldsymbol{m}|} \exp\left(-\frac{\boldsymbol{m}^2 L^2 \Gamma^2}{8} \right), \quad \Gamma \ll \omega_0, \ \Gamma^2 L \ll \omega_0.$$

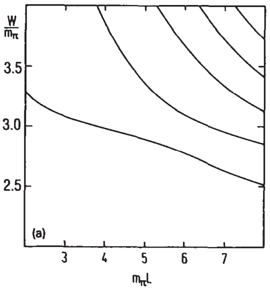
Effect of interactions on the finite-volume spectral function

- interactions shift the position of the δ-functions around
- how this happens can be studied in quantum mechanics.

Vector channel in QCD [Lüscher NPB364:237-254,1991]







Physical values of m_ρ/m_π, Γ_ρ/m_π

small departures from

$E_n = 2\sqrt{m_\pi^2 + (2\pi n/L)^2}$, $n \geq 1$

$m_\rho/m_\pi = 3$, $\Gamma_\rho/m_\pi = 0.30$

recent papers: [Jansen, Renner 1011.5288]

[PACS-CS 1011.1063, Frison et al 1011.3413]

Scattering States in a Finite Box [Lüscher, Wolff NPB 339 (1990) 222]

Consider a one-dimensional QM problem,

$$\begin{aligned} \psi(x,y) = f(x-y) &= f(y-x) \\ \{-\tfrac{1}{m}\tfrac{d^2}{dz^2} + V(|z|)\} f(z) &= E f(z)\,. \end{aligned}$$

Scattering state: for $E = k^2/m$, $k \geq 0$, choose

$$f_E(z) \stackrel{|z|\to\infty}{\sim} (1 + \ldots)\cos(k|z| + \delta(k))$$

- now consider a finite periodic box, $L \gg$ range of V
- $V_L(z) = \sum_{\nu\in\mathbb{Z}} V(|z+\nu L|)$
- in leading approx., $f_E(z)$ unchanged, but quantization condition:

$$f_E'(-\tfrac{L}{2}) = f_E'(\tfrac{L}{2}) = 0 \quad \Rightarrow \quad \boxed{\tfrac{1}{2}kL + \delta(k) = \pi n, \quad n \in \mathbb{Z}.}$$

“The kinematical phase shift kL must compensate the phase shift $2\delta(k)$ that results from the scattering to insure the periodicity of the wavefunction.”

Scattering States in a Finite Box II [Lüscher NPB364:237-254,1991]

Generalization to Quantum Field Theory:

- the Schrödinger is replaced by a Bethe-Salpeter equation, which still has asymptotic solutions $f_E(z) \sim \cos(k|z| + \delta(k))$.
- the condition $\boxed{\frac{1}{2}kL + \delta(k) = \pi n}$ still holds, where now the energy W of the two-particle state is $W = 2\sqrt{k^2 + m^2}$.

Generalization to $d = 3$:

- breaking of rotation symmetry $\Rightarrow$ infinitely many partial waves contribute
- example: two pions in a box, $W = 2\sqrt{k^2 + m_\pi^2}$
- $I^G(J^{PC}) = 1^+(1^{--})$ channel: should contain the ρ resonance
- $\ell = 1$ partial wave dominates $\Rightarrow$ phase shift determined by

$$\boxed{\phi(\tfrac{kL}{2\pi}) + \delta_{I=1,\ell=1}(k) = \pi n} \qquad n \in \mathbb{Z}; \quad \phi \text{ a known function}$$

- map out $\delta(k)$, find $L^\star$ where $\delta(k) = \frac{1}{2} \Rightarrow\ m_\rho = W^\star \equiv 2\sqrt{(k^\star)^2 + m_\pi^2}$.

Pion form factor in the time-like region

- to fully determine the spectral function, not only the finite-volume spectrum must be calculated, but also the matrix elements $\langle\pi\pi|j_\mu|0\rangle$
- how are they related to the time-like pion form factor defined in infinite-volume?,

$$\langle\pi_+\pi_-,\mathrm{out}|\boldsymbol{j}|0\rangle = e^{i\delta_1}(\boldsymbol{p}_+ - \boldsymbol{p}_-)F_\pi(Q^2)$$

- the result is [HM, in prep.]

$$|F_\pi(Q^2=M^2)|^2 = \left(q\phi'(q) + k\frac{\partial\delta_1(k)}{\partial k}\right)\frac{3\pi M^2}{k_\pi^5 L^3}|_L\langle\pi\pi|\int \mathrm{d}\boldsymbol{x}\, j^z(\boldsymbol{x})|0\rangle|^2.$$

- NB. for weakly interacting pions, $|_L\langle\pi\pi|\int \mathrm{d}\boldsymbol{x}\, j^z(\boldsymbol{x})|0\rangle|^2$ is order O(L^0)
- the proof involves introducing a fictitious photon of mass $M=\sqrt{Q^2}$
- it then follows closely the derivation of the $K\to\pi\pi$ formula by Lellouch & Lüscher hep-lat/0003023.

Real-Time Quantities in Thermal Field Theory

Thermal Field Theory

- Correlation functions: $\langle \mathrm{vac}|\phi(x)\phi(y)|\mathrm{vac}\rangle$ is generalized by

$$G(t) = \mathrm{Tr}\,\{\hat{\rho}\,\phi(t,\boldsymbol{x})\,\phi(0)\}, \qquad \hat{\rho} = \frac{e^{-\beta H}}{Z}, \quad A(t) \equiv e^{iHt}A(0)e^{-iHt}.$$

- correlators obey the Kubo-Martin-Schwinger identity $G(t) = G(t - i\beta)$
- Euclidean correlators: $G_E(t) \equiv G(-it)$, $G_E(\beta + t) = G_E(t)$
- in the Euclidean formulation of finite-temperature field theory, the time direction is compactified $0 \leq t < \beta$, $\beta = \frac{\hbar}{k_B T}$
- Euclidean frequencies are discrete: $\omega_{\mathrm{I}} = 2\pi T n, \quad n \in \mathbb{Z}$
- modified relation between Euclidean correlator and the spectral function: $C(t,\boldsymbol{k}) = \int_0^\infty d\omega\,\rho(\omega,\boldsymbol{k})\,\frac{\cosh\omega(\frac{1}{2}\beta - t)}{\sinh\frac{1}{2}\beta\omega}$
- thermal production rate of dilepton pairs of invariant mass $M^2 = \omega^2 - \boldsymbol{k}^2$:

$$\frac{\mathrm{d}N_{\ell_+\ell_-}}{\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}^3} = \left(\sum\nolimits_{\mathrm{f}} Q_{\mathrm{f}}^2\right)\frac{\alpha_{\mathrm{em}}^2}{3\pi^2}\,n_B(\omega)\,\frac{\rho_\mu^\mu(\omega,\boldsymbol{k},T)}{\omega^2 - \boldsymbol{k}^2}.$$

Transport properties: low-frequency limit of the spectral function

- the relaxation to equilibrium of long wavelength fluctuations are described by the small-ω behavior of various spectral functions
- these are associated with correlation functions of the conserved currents (energy, momentum, particle number)

Quantities Excited	**Transport Coefficient**
quark number (u,d,s,c..)	diffusion constant D: $\sim \exp -Dk^2 t$
transverse momentum	shear viscosity η: $\sim \exp -\dfrac{\eta k^2 t}{e+p}$
energy, longit. momentum, pressure	shear + bulk viscosity: $\sim \exp -\frac{1}{2}\Gamma_s t$,
(sound waves)	$\Gamma_s = \frac{4}{3}\eta + \zeta$

the diffusion coefficient D is related to the current correlator,

$$D\chi_s = \lim_{\omega\to 0}\lim_{\boldsymbol{k}\to 0} \frac{\pi}{\omega}\rho_{jj}(\omega, \mathbf{k})$$ Kubo formula

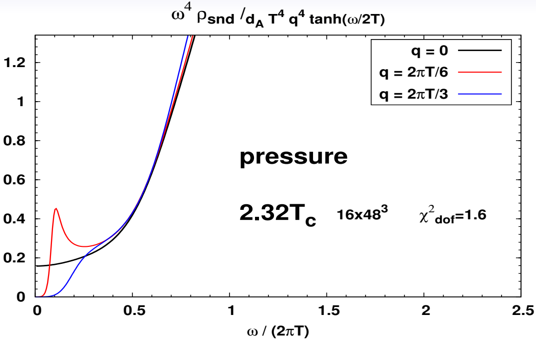
the shear viscosity η is related to the correlator of shear stress T_{xy}

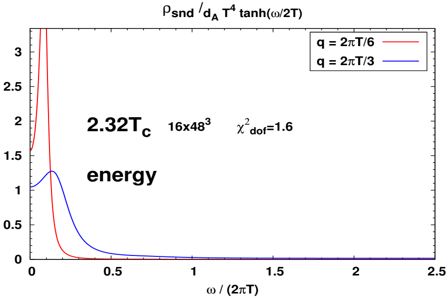
$$\eta = \lim_{\omega\to 0}\lim_{\boldsymbol{k}\to 0} \frac{\pi}{\omega}\rho_{xy,xy}(\omega, \mathbf{k})$$ Kubo formula

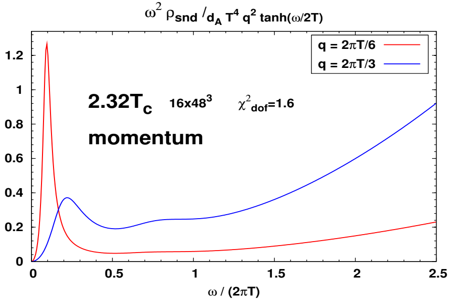
Sound channel spectral functions from 16×48^3 lattice [HM, QM 09]

- 48 data points
- 7 fit parameters
- highest momentum included: $q = \pi T$
- smallest t included: $t/a = 4$
- educated guess for viscosity at LHC:

$$[\eta/s]_{QGP} \approx [\eta/s]_{GP,lat} \cdot \left[\frac{[\eta/s]_{QGP}}{[\eta/s]_{GP}} \right]_{AMY} \approx 0.40$$

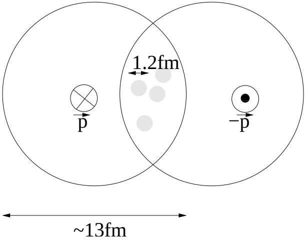






Heavy ion collisions and elliptic flow

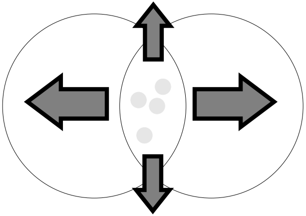
Transverse section of the collision of two gold nuclei:



elliptic flow coefficient:

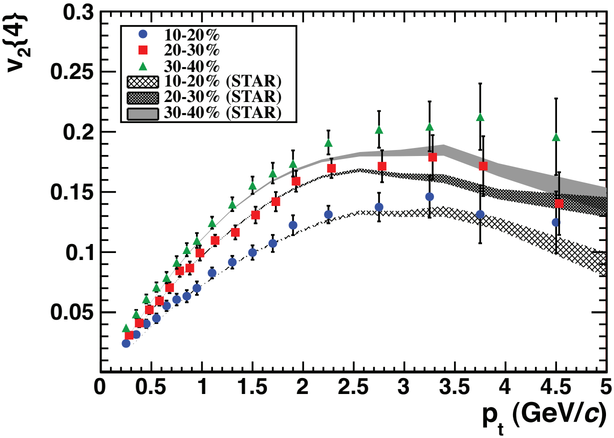
$$v_2(p_T) = \langle \cos(2\phi) \rangle_{p_T} \equiv \frac{\int_{-\pi}^{\pi} d\phi \cos(2\phi) \frac{d^3N}{dy\, p_t dp_t\, d\phi}}{\int_{-\pi}^{\pi} \frac{d^3N}{dy\, p_t dp_t\, d\phi}}$$

Interpretation:



- $\vec{b}$= impact vector
- pressure gradient is greater in the $\vec{b}$ direction
- $\Rightarrow$ excess of particles produced in that direction.

Elliptic flow: from RHIC to LHC



- Pb-Pb collisions at $\sqrt{s}/A = 2.76$TeV vs. Au-Au collisions at $\sqrt{s}/A = 200$GeV.

[STAR Nucl. Phys. A 757 (2005) 102; BRAHMS Nucl. Phys. A 757 (2005) 1;
PHOBOS Nucl. Phys. A 757 (2005) 28; PHENIX Nucl. Phys. A 757 (2005) 184; ALICE 1011.3914]

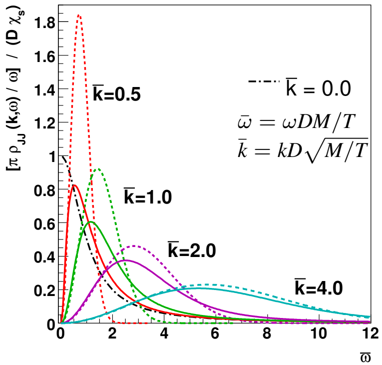
Heavy quarks in heavy-ion collisions

- measurements at RHIC have shown that heavy quarks display substantial elliptic flow [STAR nucl-ex/0607012, PHENIX nucl-ex/0611018]
- $\Rightarrow$ stronger medium interactions than perturbation theory would suggest
- assuming the charm quark is 'heavy', the RHIC data requires a **momentum diffusion coefficient** $\boxed{\kappa/T^3 \gtrsim 2}$ (κ defined on next slide; [Teaney, Casalderrey-Solana hep-ph/0605199])
- in perturbation theory to strict leading order, κ is given by

$$\kappa = \frac{g^2 C_F T}{6\pi} m_D^2 \left(\log \frac{2T}{m_D} + \frac{1}{2} - \gamma_E + \frac{\zeta'(2)}{\zeta(2)} + \frac{N_\mathrm{f} \log 2}{2N_c + N_\mathrm{f}} \right)$$ [Moore, Teaney hep-ph/0412346]

- for realistic values of the Debye screening length $m_D = O(gT)$, $\kappa < 0$
- non-perturbative methods are needed to predict κ with confidence.

Langevin description and heavy-quark current spectral function



Description through a Langevin equation:

[Teaney, Petreczky hep-ph/0507318]

$$\frac{d\boldsymbol{x}}{dt} = \frac{\boldsymbol{p}}{M},$$
$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{\xi}(t) - \eta\, \boldsymbol{p}(t),$$
$$\langle \xi^i(t)\xi^j(t')\rangle = \boldsymbol{\kappa}\, \delta^{ij}\delta(t-t').$$

$\leftarrow$ leads to a prediction for the spectral function of the heavy-quark current

- Fluctuation-dissipation relation (Einstein 1905): $\boxed{\eta = \frac{\kappa}{2MT}.}$
- at late times, the Langevin equation describes diffusion, $\langle x^2(t)\rangle \sim 2Dt$, with a diffusion coefficient $\boxed{D = 2T^2/\kappa.}$

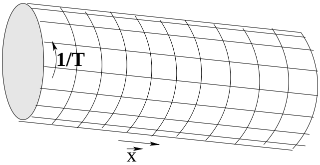
From the Langevin equation to Heavy-Quark Effective Theory

$$\langle \xi^i(t)\xi^j(t')\rangle \;=\; \boldsymbol{\kappa}\, \delta^{ij}\delta(t-t').$$

Since $g\boldsymbol{E}$ is the color-Lorentz force acting on the heavy-quark, it is not surprising that in QCD, κ can be extracted from an electric-field correlator,

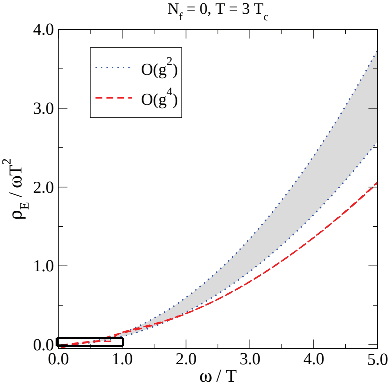
$$G_E^{\rm HQET}(t) = \frac{\left\langle \mathrm{Re\,Tr}\left(U(\beta,t)gE_k(t,\mathbf{0})U(t,\mathbf{0})gE_k(0,\mathbf{0})\right)\right\rangle}{-3\,\langle \mathrm{Re\,Tr}\,U(\beta,0)\rangle}, \qquad \kappa = \lim_{\omega\to 0}\frac{2\pi T}{\omega}\rho^{\rm HQET}(\omega)$$

- the color parallel transporters $U(t_2,t_1)$ in the fundamental representation are propagators of static quarks
- the Polyakov loop appears in the denominator.

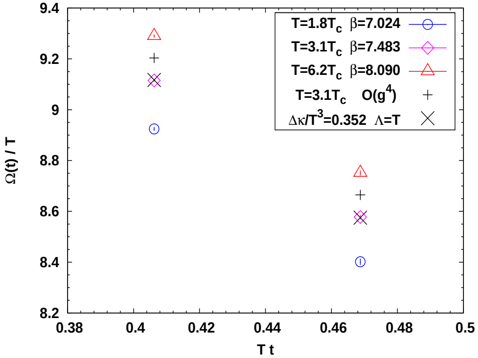


Casalderrey-Solana, Teaney hep-ph/0605199; Caron-Huot, Laine, Moore 0901.1195

$$\int_0^\infty d\omega\, \rho(\omega, \boldsymbol{k})\, \frac{\cosh \omega(\frac{1}{2}\beta - t)}{\sinh \frac{1}{2}\beta\omega} \qquad = \qquad G_E(t, \boldsymbol{k})$$



$$\frac{G_E(t-a/2)}{G_E(t+a/2)} = \frac{\cosh[\Omega(t)(\beta/2-(t-a/2))]}{\cosh[\Omega(t)(\beta/2-(t+a/2))]}$$



- a correction $\Delta\rho(\omega) = \frac{1}{\pi}\Delta\kappa \tanh(\omega/2T)\theta(\Lambda - |\omega|)$ to the O(g^4) result can fit the Euclidean data
- this represents a large correction for κ/T^3 from about 0.23 to 0.58.

[Burnier et al. 1006.0867, HM 1012.0234]

Final Remarks

- ‘reconstructing’ the spectral function from Euclidean correlation functions is a general (and hard) problem of numerical quantum field theory

- so far, progress has mainly come from understanding the physics of the problem better

- significant progress in formulating the problem of a diffusing heavy-quark non-perturbatively, and first lattice calculation

- open **?**: is there an analogue of the Lüscher formula, by which transport coefficients could be extracted?

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