Moduli spaces of cold holographic matter

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Outline

1. Introduction
   - Motivation
   - Moduli spaces
   - Results
2. Technique
   - D3/D7
   - D3/D5
3. Conclusions
   - Summary
   - Future directions
   - Speculation

Motivation

Holography

Duality between String Theory in d+1 dimensions and Quantum Field Theory in d-dimensions

Why are we using holography?

- Different way of thinking about quantum systems: new understanding;
- Strongly coupled quantum systems are hard to describe with standard methods.

Two different approaches

- bottom-up;
- top-down.

Deconstructing the title

Cold holographic matter

Cold: zero temperature

Holographic matter: compressible states from holography

Compressible states

- charge density smooth with chemical potential
- turn on gauge field component A_t

Examples in nature

- superfluids
- solids
- fermi liquids

Moduli spaces

We study **moduli spaces of cold holographic matter**

What is a moduli space?

Geometric space in which points are objects of a certain kind

Instanton moduli space

- each point of the manifold is a different solution to the self dual equations

Moduli space of vacua (in a field theory)

- each point is a possible ground state

String theory connects them

Moduli space of vacua

Consider $\mathcal{N}=4$, $SU(N_c)$ gauge theory with $\mathcal{N}=2$ matter

- $\mathcal{N}=4$ vector multiplet
  (contains $\Phi_1,\ \Phi_2,\ \Phi_3 \longrightarrow N_c \times N_c$ matrices)
- $N_f \ll N_c$ fundamental hypermultiplets of $\mathcal{N}=2$
  (contain $\tilde{Q}_i,\ Q^i \longrightarrow N_c$-legged vectors, $i=1,\ldots,N_f$)

Superpotential is

$$W = \tilde{Q}_i \Phi_3 Q^i + \mathrm{Tr}(\epsilon_{IJK}\Phi_I \Phi_J \Phi_K)$$

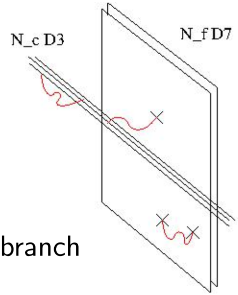
Minimize it!

- Coulomb branch:
  $Q^i = 0 = \tilde{Q}_i, \qquad \Phi_1, \Phi_2, \Phi_3$ mutually commuting
- Higgs branch:
  $Q^i, \tilde{Q}_i$ nonzero, $\qquad \Phi_3 = 0$

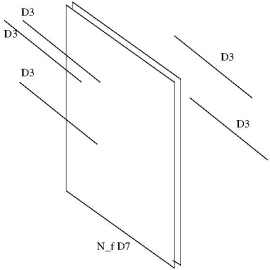
Moduli space of vacua - brane perspective

Gauge theory content

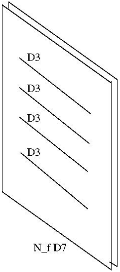
- Φ_1, Φ_2, Φ_3 from 3-3 strings
- $\tilde{Q}_i$, Q^i from 3-7 strings



Coulomb branch

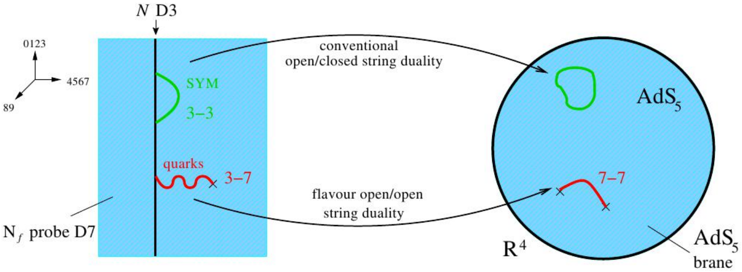


Higgs branch



Probe branes

Probe approximation ($N_c \gg N_f$) can be summarized as



[Erdmenger, Evans, Kirsch, Threlfall '07]

In probe approximation

$$S = S_{D7} + T_7 \int P[C_4] \wedge F \wedge F$$

Instanton on $D7 \longrightarrow$ sources $\int P[C_4]$ as if it was a $D3$

Higgs branch $\equiv$ instanton moduli space

What we have done

We have studied the Higgs branch in 4ND probe systems, with $N_f = 1$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	X	X	X	X						
D7	X	X	X	X	X	X	X	X		

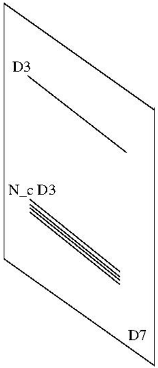
	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	X	X	X	X						
D5	X	X	X		X	X	X			

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	X	X	X	X						
D3′	X	X			X	X				

Probe branes

We'll always assume

- $N_c \gg N_f$ (probe approximation);
- $N_f = 1$
- $D3$ leave stack in small numbers: $N_c - 1 \sim N_c$ (background unchanged)
- non zero baryon density, zero temperature: cold holographic matter



Findings

It is known that

- gauge theory dual to D3/D7 has no moduli space of vacua (when $N_f = 1$)
- gauge theories dual to D3/D5 and D3/D3 have a moduli space

We will see that, with non zero baryon density

- moduli space emerges in dual to D3/D7
- moduli space still exists in duals to D3/D5 [Chang, Karch '12] and D3/D3

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The D3 background

Consider a stack of N_c D3s

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	X	X	X	X						

Low energy physics is $\mathcal{N} = 4$ SYM with $SU(N_c)$ gauge group.
Describe this brane using holography

- $N_c \to \infty$, $g_{YM} \to 0$, $\lambda = g_{YM}^2 N_c$ fixed
- $\lambda \to \infty$

The background is ($R^4 = 4\pi g_s N_c \alpha'^2$)

$$ds^2 = Z^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}\left(dr^2 + r^2 ds^2_{\mathbb{S}^5}\right), \qquad Z(r) \equiv R^4/r^4,$$

$$F_5 = \frac{4}{R}\left(\mathrm{vol}_{AdS_5} + \mathrm{vol}_{\mathbb{S}^5}\right), \qquad F_5 = dC_4$$

D7 probe

We probe this system with a D7

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	X	X	X	X						
D7	X	X	X	X	X	X	X	X		

The low energy description is $\mathcal{N}=4$ SYM with gauge group $SU(N_c)$, together with $\mathcal{N}=2$ fundamental matter.
Dual is D7 in the D3 background

$$S_7 = -T_7 \int d^8\xi \sqrt{\det(-P[G]_{ab} + F_{ab})} + \frac{1}{2} T_7 \int P[C_4] \wedge F \wedge F.$$

Ansatz

$$A(\xi) = A_0(z) dx^0 + A_i(z) dz^i, \qquad (z^1, z^2, z^3, z^4) = (x^4, x^5, x^6, x^7)$$

D7 probe

Recall: what we will show is the existence of a moduli space

On shell action the same for different configurations

No dependence on the field theory coordinates

- integrate them

Write the action as an action in 4d (z^1, z^2, z^3, z^4)

$$s_7 = -T_7 \int d^4 z \left[\sqrt{\det(g_{ij} + Z^{-1/2} f_{ij})} - \frac{1}{8} Z^{-1} \tilde{\epsilon}^{ijkl} f_{ij} f_{kl} \right],$$

g_{ij} is an effective metric. [Chen, Hashimoto, Matsuura '09]

$$g_{ij} \equiv \delta_{ij} - \partial_i A_0 \partial_j A_0, \qquad f_{ij} \equiv \partial_i A_j - \partial_j A_i.$$

The most important formula of the presentation

For general 4×4 symmetric $\mathcal{G}$ and antisymmetric $\mathcal{F}$

$$\sqrt{\det\left(\mathcal{G}_{ij} + \mathcal{F}_{ij}\right)} \geq \sqrt{\det \mathcal{G}_{ij}} + \frac{1}{8}\left|\tilde{\epsilon}^{ijkl}\mathcal{F}_{ij}\mathcal{F}_{kl}\right|$$

[Gibbons, Hashimoto '00]

- Inequality saturated for self dual $\mathcal{F}$ with respect to metric $\mathcal{G}$

$$\mathcal{F}_{ij} = \pm\frac{1}{2}\epsilon_{ijkl}\mathcal{F}^{kl}, \qquad \epsilon^{ijkl} \equiv \tilde{\epsilon}^{ijkl}/\sqrt{\det \mathcal{G}_{ij}}.$$

- uses the fact that $\mathcal{G}$ and $\mathcal{F}$ are 4×4 matrices: will be useful for 4ND systems
- useless for non $4ND$ systems. No moduli spaces there.

Next step

Use this to show the existence of degenerate ground states: moduli space!

Back to our setting

Our action is reduced to

$$s_7 \leq -T_7 \int d^4z \left[\sqrt{\det g_{ij}} + \frac{1}{8} Z^{-1} \left(|\tilde{\epsilon}^{ijkl} f_{ij} f_{kl}| - \tilde{\epsilon}^{ijkl} f_{ij} f_{kl} \right) \right].$$

- bound saturated for self dual f wrt effective metric g
- self dual f extremizes action $\longrightarrow$ solves equation of motion
- self dual f maximizes action $\longrightarrow$ minimizes Helmholtz free energy

$$s_7 = -T_7 \int d^4z \sqrt{\det g_{ij}},$$

Recall: $g_{ij} \equiv \delta_{ij} - \partial_i A_0 \partial_j A_0$

Interpretation

$\int F \wedge F$ is a number $\longrightarrow$ D3 brane charge in the D7, sources $\int P[C_4]$

Recipe

f_{ij} out of the action is very convenient

Makes it very easy to get solutions

To get explicit solutions

- solve electrostatic problem. Get $A_0(z)$
- $A_0(z)$ defines the effective metric
- get $A_i(z)$ from self dual equations

In particular, vacuum energy independent of f_{ij} (as it should...)

But, don't forget

The moduli space exists only if we find normalizable solutions

Example

The purpose of the example

- Show you why there is no moduli space at zero density
- Show you what happens when we turn on the density

Possible $A_0(z)$ that describes a compressible state is

$$A_0'(\rho) = \frac{1}{\sqrt{1 + \rho^6/\rho_0^6}}\,, \qquad \rho_0^6 \text{ related to density}$$

- ρ is the radial coordinate (boundary at $\rho \to \infty$)
- solves equations of motion

Effective metric is then

$$g_{ij}\, dz^i dz^j = \frac{\rho^6}{\rho^6 + \rho_0^6}\, d\rho^2 + \rho^2 ds^2_{\mathbb{S}^3}\,.$$

Effective metric

$$g_{ij}\, dz^i dz^j = \frac{\rho^6}{\rho^6 + \rho_0^6}\, d\rho^2 + \rho^2 ds^2_{\mathbb{S}^3} \,.$$

- Metric is conformally flat

$$g_{ij}\, dz^i dz^j = \Omega(\bar\rho)^2 \left(d\bar\rho^2 + \bar\rho^2 ds^2_{\mathbb{S}^3}\right) \,, \qquad \Omega(\bar\rho) = \left(1 - \rho_0^6/(4\bar\rho^6)\right)^{1/3}$$

- Range of new coordinate is $\bar\rho \in [2^{-1/3}\rho_0, \infty)$

Solve self dual equation in this background

$$f_{ij} = \frac{1}{2}\epsilon_{ijkl} f^{kl}$$

- it is conformal $\longrightarrow$ solve it in flat space
- range of radius is $\bar\rho \in [2^{-1/3}\rho_0, \infty)$

Effectively

Non zero density is the same as zero density, but with ball excised

Solution is

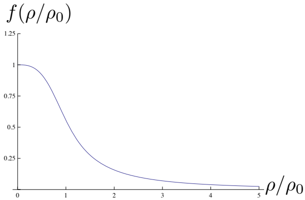
$$A_\alpha(z) = \sum_{l=1}^{\infty} \left(b_l\, \bar\rho^{l+1} \mathcal{Y}_\alpha^{l,+} + \frac{c_l}{\bar\rho^{l+1}} \mathcal{Y}_\alpha^{l,-} \right), \qquad A_{\bar\rho}(z) = 0\,,$$

- Normalizable solutions: $b_l = 0$ (no sources, other than chemical potential)

Recall,

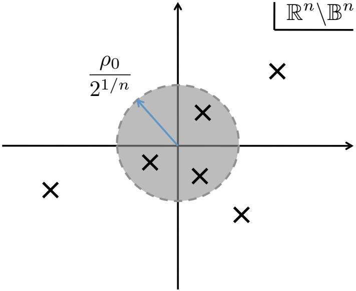
- zero density: $\bar\rho = \rho \in [0, \infty)$
- non zero density: $\bar\rho(\rho) \in [2^{-1/3}\rho_0, \infty)$

$$A_\alpha(z) = c\, \frac{2^{2/3}}{\rho_0^2} \left[\frac{\sqrt{1+\rho_0^6/\rho^6} - 1}{\sqrt{1+\rho_0^6/\rho^6} + 1} \right]^{1/3} \mathcal{Y}_\alpha^{1,-}$$



Infinite series changes position of singularity → multipole expansion

A pictorial way to see what is going on is



Non zero density de-singularizes the solution

Summing up

- no moduli space at zero density
- non-singular solutions at nonzero density: moduli space

Meaning of the c_ls

In the field theory, c_ls are related with expectation values of operators

$$\langle \mathcal{O}_l^- \rangle \sim c_l, \qquad O_l^- = Q^\dagger \sigma^I \chi Q$$

(χ: product of $l-1$ symmetrized adjoint scalars)
The instanton number is

$$\int F \wedge F \sim \text{sum of } c_l^2$$

Also, it is the number of D3s dissolved into D7s

$$S_{WZ} = T_7 \int P[C_4] \wedge F \wedge F$$

Should be quantized?

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The D3 background - déjà vu

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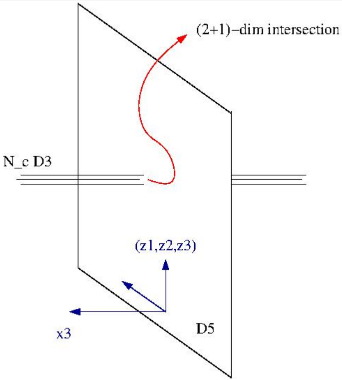
$$F_5 = \frac{4}{R}\left(\mathrm{vol}_{AdS_5} + \mathrm{vol}_{\mathbb{S}^5}\right), \qquad F_5 = dC_4$$

Probe D5

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	X	X	X	X						
D5	X	X	X		X	X	X			



Low energy description has

- 4d $\mathcal{N} = 4$ SYM with gauge group $SU(N_c)$
- 3d $\mathcal{N} = 4$ fundamental hypermultiplets living in the intersection

Dual is D5 in the D3 background

$$S_7 = -T_5 \int d^6\xi \sqrt{\det(-P[G]_{ab} + F_{ab})} + T_5 \int P[C_4] \wedge F.$$

Ansatz relabel $(z^1, z^2, z^3) = (x^4, x^5, x^6)$

$$x^3(\xi) = x^3(z), \qquad A(\xi) = A_0(z)dx^0 + A_i(z)dz^i$$

Probe D5

No dependence in field theory coordinates

- integrate them

We are lead to

$$s_5 = -T_5 \int d^3 z \left[\sqrt{\det(g_{ij} + Z^{-1/2} f_{ij} + Z^{-1} \partial_i x^3 \partial_j x^3)} - \frac{1}{2} Z^{-1} \tilde{\epsilon}^{ijk} \partial_i x^3 f_{jk} \right] ,$$

g_{ij} is a 3d effective metric, and f_{ij} a field strength in $\mathbb{R}^3$

$$g_{ij} \equiv \delta_{ij} - \partial_i A_0 \partial_j A_0, \qquad f_{ij} \equiv \partial_i A_j - \partial_j A_i .$$

Recall

We have done something very similar before. We wrote the action as in 4d, and used an inequality.

Cosmetic uplift

Recall: we will use at some point the inequality

For general symmetric $\mathcal{G}$ and antisymmetric $\mathcal{F}$

$$\sqrt{\det\left(\mathcal{G}_{ij}+\mathcal{F}_{ij}\right)} \geq \sqrt{\det \mathcal{G}_{ij}} + \frac{1}{8}\left|\tilde{\epsilon}^{ijkl}\mathcal{F}_{ij}\mathcal{F}_{kl}\right| .$$

We need to write the action in a more $4d$ fashion.

- introduce extra direction z^4
- uplift the fields to $\mathbb{R}^4$

$$\hat{g}_{ij} = g_{ij} + \delta^4{}_i\,\delta^4{}_j\,, \qquad \hat{a} = A_i(z)dz^i + x^3(z)dz^4\,, \qquad \hat{f} = d\hat{a},$$

- demand they don't depend on z^4

We get to

$$s_5 = -T_5\int d^4z\left[\sqrt{\det(\hat{g}_{ij}+Z^{-1/2}\hat{f}_{ij})} - \frac{1}{8}Z^{-1}\tilde{\epsilon}^{ijkl}\hat{f}_{ij}\hat{f}_{kl}\right],$$

Looks very much like previous D3/D7 action (reason: T-duality)

Using the inequality, we get

$$s_5 \leq -T_5 \int d^4 z \left[\sqrt{\det \hat{g}_{ij}} + \frac{1}{8} Z^{-1} \left(|\tilde{\epsilon}^{ijkl} \hat{f}_{ij} \hat{f}_{kl}| - \tilde{\epsilon}^{ijkl} \hat{f}_{ij} \hat{f}_{kl} \right) \right].$$

Bound is saturated for self dual $\hat{f}$ wrt $\hat{g}$

$$s_5 = -T_5 \int d^3 z \sqrt{\det g_{ij}},$$

- independent of $A_i(z)$ and $x^3(z)$

Self dual equation in terms of original variables is vector/scalar duality

$$\partial_i x^3 = \frac{1}{2} \epsilon_{ijk} f^{jk},$$

All very similar with D3/D7

We'll have a moduli space IF we find normalizable solutions

We go throw the same procedure

- solve electrostatic problem: find $A_0(z)$
- find solution for vector scalar duality equation

Possible $A_0(z)$ for a compressible state is

$$A_0'(\rho) = \frac{1}{\sqrt{1+\rho^4/\rho_0^4}}, \qquad \rho_0^4 \text{ related to density}$$

The effective metric is now

$$g_{ij}dz^i dz^j = \frac{\rho^4}{\rho^4+\rho_0^4}d\rho^2 + \rho^2 ds^2_{\mathbb{S}^2}.$$

- metric is again conformal, but...
- cannot solve equation in flat space: equation is not conformally invariant

Solving the vector/scalar dual equation

Form notation

We want to solve $dx^3 = \star F$

In particular, it is true

$$d \star dx^3 = dF$$

If there are no sources

- $dF = 0$ is the Bianchi identity
- both $A_i(z)$ and $x^3(z)$ are harmonic ($d \star d = 0$)

If there are sources

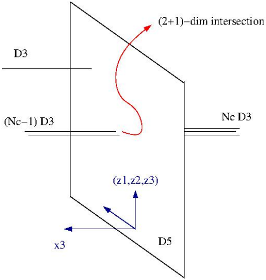
- monopole source gives $dF \neq 0$ and $\int F \neq 0$
- WZ term tells us it is D3 brane charge within the D5

$$T_5 \int P[C_4] \wedge F$$

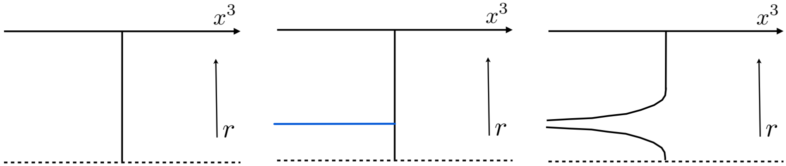
Zero density

Solutions ($l = 0$) are of the form

$$x^3(z) = \sum_m \frac{C_m}{|z - z'_m|}, \quad \text{and corresponding F}$$



- they are singular
- they have a physical interpretation



Punchline

At zero density, D3/D5 has a moduli space! (already known...)

The field theory side

Field theory contains two scalar spin-l operators $\mathcal{O}_l^{\pm}$, dual to

$$\varphi_l^+(\rho) \equiv l\,A_l(\rho) + \rho\,x_l^3(\rho)$$
$$\varphi_l^-(\rho) \equiv (l+1)A_l(\rho) - \rho\,x_l^3(\rho)$$

Operators $\mathcal{O}_l^{\pm}$ are a sandwich of squarks, with symmetrized adjoints in the middle

For the zero density:

- $\varphi_l^+(\rho) = 0$ and $\langle\mathcal{O}_l^+\rangle = 0$
- $\langle\mathcal{O}_l^-\rangle \neq 0$ in general

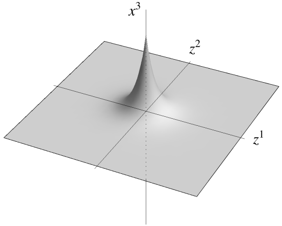
Non zero density

Effective metric is different, hence solutions will be different

- still exist these spiky solutions
- also new solutions with kink in the origin

Properties of these solutions

- Depend on a parameter c_l
- $\varphi_l^{\pm}(\rho) \neq 0$ in general
- $\langle \mathcal{O}_l^{+} \rangle \propto c_l$ and $\langle \mathcal{O}_l^{-} \rangle \propto c_l$



So...

Moduli space survives the introduction of density

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Summary

- We studied D3/Dp systems ($p = 3, 5, 7$), with 4ND directions at non zero density
- We constructed normalizable solutions with the same free energy
  - D3/D7: excising a ball desingularizes solution
  - D3/D5: new kinky, non spiky, solutions
- Dual scalar operators get VEVs that parametrize the Higgs branch

It is very surprising that there is a Higgs branch at non zero density. Large-N artifact?

Future directions

- Is there any relation between effective metric and moduli space metric?
- Turn on T, turn on B. Do they lift the moduli space?
- Are there states with lower energy? (eg. Multi BIon)

Speculation

Can we classify holographic matter? From observations:

	4ND	6ND	8ND
Higgs branch	X		
Striped instability		X	
SUSY			X: $(0+1)$d fermion
Chiral anomaly	X: D4's in ABJM		X: $(1+1)$d fermion

Correlation between number of ND's and character of compressible state

- related with form of WZ terms
- are these just special cases?

Can anomalies classify holographic matter? K-theory?