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April 18, 2013

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Our calculation

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- We will calculate the motion of a heavy quark in a magnetic field at zero temperature.
  This will be done through the AdS/CFT correspondence by considering a string moving in pure AdS_5 spacetime.

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- We will calculate the motion of a heavy quark in a magnetic field at zero temperature.
  This will be done through the AdS/CFT correspondence by considering a string moving in pure AdS_5 spacetime.
- The charge can be a flavor charge (electric charge is a special case of this), and the magnetic field should be thought as being imposed on the flavor brane. Here we will treat this by imposing the magnetic field at the endpoint of the string at $r = \Lambda$ in pure AdS with metric $ds^2 = \frac{L^2}{r^2}\left(-dt^2 + dr^2 + dx^2 + dy^2 + dz^2\right)$.

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- Magnetic fields induce the chiral magnetic effect in strongly coupled matter, and this may have implications both for heavy ion experiments as well as neutron stars.

Motivation of the calculation

Motion in a magnetic field at zero temperature

- Magnetic fields induce the chiral magnetic effect in strongly coupled matter, and this may have implications both for heavy ion experiments as well as neutron stars.
- Magnetic fields are also one of the most important environments in condensed matter experiments. The Hall conductivity is one of the main observables that can be calculated in this way.

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- We consider the case of AdS_5 as the background of the bulk, because in that case we have analytic solutions that connect the motion of the string with the motion of the endpoint living on the boundary of AdS (solutions obtained by Mikhailov).

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- We consider the case of AdS_5 as the background of the bulk, because in that case we have analytic solutions that connect the motion of the string with the motion of the endpoint living on the boundary of AdS (solutions obtained by Mikhailov).
- In that case $X^{\mu}(\tau, r) = \tilde{x}^{\mu}(\tilde{\tau}) \pm r\frac{d\tilde{x}^{\mu}}{d\tilde{\tau}}$, with r the radial coordinate in the Poincare patch, $\tilde{\tau}$ the proper time at the boundary at $r = 0$ and $\tilde{x}^{\mu}$ the coordinates of the endpoint at the boundary at $r = 0$.

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- The solutions with the $+$ sign are the retarded ones, which we will consider.

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- The solutions of Mikhailov have the intepretation in global AdS_5 space that the fluctuations of the string in one brane are absorbed completely by the anti-brane.

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- The solutions with the $+$ sign are the retarded ones, which we will consider.
- The solutions of Mikhailov have the intepretation in global AdS_5 space that the fluctuations of the string in one brane are absorbed completely by the anti-brane.
- We will consider a flavour brane at $r = \Lambda$ and the motion of the endpoint corresponds to the motion of a quark with mass $m_q = \frac{\sqrt{\lambda}}{2\pi\Lambda}$ with $\lambda = g^2_{YM}N_c$ the t′ Hooft coupling.

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- Then the solutions of Mikhailov read:

$$X^\mu(\tau, r) = \left(\frac{r - \Lambda}{\sqrt{1 - \Lambda^4 \frac{4\pi^2}{\lambda}\mathcal{F}^2}}\right)\left(\frac{dx^\mu}{d\tau} - \frac{2\pi}{\sqrt{\lambda}}\Lambda^2 \mathcal{F}^\mu\right) + x^\mu(\tau),$$

with τ, x^μ the proper time and the coordinates of the endpoint at $r = \Lambda$, the position of the flavour brane and $\mathcal{F}^\mu$ the 4-force on the quark.

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  with τ, x^{μ} the proper time and the coordinates of the endpoint at $r = \Lambda$, the position of the flavour brane and $\mathcal{F}^{\mu}$ the 4-force on the quark.

- We consider a constant magnetic field on the boundary $\vec{B} = B\hat{z}$ and we request the variation of the action

$$S = -\frac{1}{2\pi\ell_s^2} \int dr\, d\tau \sqrt{-detg} + e \int d\tau A^{\mu} \frac{dx^{\mu}}{d\tau}$$

  on the boundary at $r = \Lambda$ to be zero. The bulk variation is automatically zero for the solutions of Mikhailov in the case of pure AdS_5.

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Equations of motion for the endpoint

- When an external force $\mathcal{F}^\mu$ is exerted on the quark, its equation of motion reads:

$$\frac{d}{d\tau}\left(\frac{m\frac{dx^\mu}{d\tau} - \frac{\sqrt{\lambda}}{2\pi m}\mathcal{F}^\mu}{\sqrt{1-\frac{\lambda}{4\pi^2 m^4}\mathcal{F}^2}}\right) = \frac{\mathcal{F}^\mu - \frac{\sqrt{\lambda}}{2\pi m^2}\mathcal{F}^2\frac{dx^\mu}{d\tau}}{1-\frac{\lambda}{4\pi^2 m^4}\mathcal{F}^2}$$

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- When the external force is due to the magnetic field then the force components become

$$\mathcal{F}^0 = 0 \quad , \quad \mathcal{F}^x(\tau) = -B\dot{y}(\tau) \quad , \quad \mathcal{F}^y(\tau) = B\dot{x}(\tau)$$

and its equation of motion reads:

$$\ddot{\vec{x}} = -\frac{s\sqrt{1-(\gamma^2-1)s^2}}{\gamma(1+s^2)}\left(s\dot{\vec{x}} + \hat{z}\times\dot{\vec{x}}\right),$$

where we have considered $\Lambda = 1, B = s\frac{m}{\Lambda}$.

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- When the external force is due to the magnetic field then the force components become

$$\mathcal{F}^0=0 \quad , \quad \mathcal{F}^x(\tau)=-B\dot{y}(\tau) \quad , \quad \mathcal{F}^y(\tau)=B\dot{x}(\tau)$$

and its equation of motion reads:

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where we have considered $\Lambda=1, B=s\frac{m}{\Lambda}$.

- The maximum initial velocity is given by $v_{max}=\frac{1}{\sqrt{1+s^2}}$.

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- The motion can't be found analytically, apart from the case of small initial velocity compared to v_{max}, i.e. $v_0 << v_{max}$. In this case the motion is

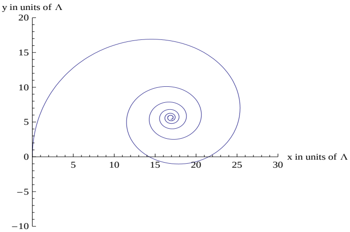
$$\begin{pmatrix} x^{(0)}(t) \\ y^{(0)}(t) \end{pmatrix} = R_0 e^{-\frac{s^2}{1+s^2}t} \begin{pmatrix} \cos(\frac{s}{1+s^2}t + \phi_0) \\ \sin(\frac{s}{1+s^2}t + \phi_0) \end{pmatrix} + \begin{pmatrix} A \\ B \end{pmatrix}.$$

Trajectory of the quark
The motion of the quark

- The velocity of the quark is

$$v^2(t) = \frac{sech^2\left(\frac{s^2 t}{s^2+1} + \tanh^{-1}\left(\sqrt{1-(s^2+1)\,v_0^2}\right)\right)}{s^2+1}$$

- In the picture we see the motion for small s. For large s the motion ends quickly without forming a spiral.



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- The four-momentum of the quark and the rate at which four-momentum is carried away are

$$\frac{dE_{rad}}{dt} = \frac{m s^2 \dot{\vec{x}}(t)^2}{\sqrt{1-\dot{\vec{x}}(t)^2}\left(1-\left(s^2+1\right)\dot{\vec{x}}(t)^2\right)}$$

$$\frac{d\vec{P}_{rad}}{dt} = \left(\dot{\vec{x}}(t) + s\dot{\vec{x}}(t)\times\hat{z}\right)\frac{dE_{rad}}{dt}.$$

- The bulk and boundary actions are time independent and therefore the total energy is conserved. This means that the energy lost by the quark is stored in the gluonic degrees of fredom and propagates to infinity.

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- The bulk and boundary actions are time independent and therefore the total energy is conserved. This means that the energy lost by the quark is stored in the gluonic degrees of fredom and propagates to infinity.
- We note that the rate of energy transfer $\frac{dE_{rad}}{dt}$ is an increasing function of $||\dot{\vec{x}}||$ which means that the faster the particle moves, the faster is the rate at which it loses its energy.

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- The bulk and boundary actions are time independent and therefore the total energy is conserved. This means that the energy lost by the quark is stored in the gluonic degrees of fredom and propagates to infinity.
- We note that the rate of energy transfer $\frac{dE_{rad}}{dt}$ is an increasing function of $||\dot{\vec{x}}||$ which means that the faster the particle moves, the faster is the rate at which it loses its energy.
- Then we obtain

$$\frac{dE_{rad}}{dt} = \frac{ms^2 csch^2\left(\frac{s^2 t}{s^2+1} + \tanh^{-1}\left(\sqrt{1-(s^2+1)\,v_0^2}\right)\right)}{\sqrt{(s^2+1)\left(sech^2\left(\frac{s^2 t}{s^2+1} + \tanh^{-1}\left(\sqrt{1-(s^2+1)\,v_0^2}\right)\right) + s^2+1\right)}}$$

and we observe that for late times $t >> \frac{s^1+1}{s^2}$

$$\frac{dE_{rad}}{dt} \propto e^{-\frac{2s^2 t}{1+s^2}},$$

has an exponential damping as the squared velocity $v(t)^2$.

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Horizon of the induced metric

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- The induced metric elements are

$$g_{\tilde{\tau}\tilde{\tau}} = -\frac{L^2}{r^2}\left(1 - r^2\ddot{\tilde{x}}^{\mu}(\tilde{\tau})\ddot{\tilde{x}}_{\mu}(\tilde{\tau})\right), g_{\tilde{\tau}r} = -\frac{L^2}{r^2} \quad , \quad g_{rr} = 0$$

where $\tilde{x}^{\mu} = \{\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}\}$ and $\tilde{\tau}$ is the proper time on the auxiliary boundary at $r = 0$.
We will find the position of the horizon with time in the approximation where the initial velocity v_0 of the endpoint at $r = \Lambda$ is very small $v_0 << v_{max} = \frac{1}{\sqrt{1+s^2}}$.

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We will find the position of the horizon with time in the approximation where the initial velocity v_0 of the endpoint at $r = \Lambda$ is very small $v_0 << v_{max} = \frac{1}{\sqrt{1+s^2}}$.
- The equations for the evolution of light rays are given by

$$\begin{aligned} dt &= 0 \\ \frac{dr}{dt} &= -\left(1 - r^2\frac{s^2 v(t)^2}{1 - v(t)^2}\right)\sqrt{\frac{1 - v(t)^2}{1 - \frac{s^2 v(t)^2}{1 - v(t)^2}}} \end{aligned}$$

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- For $v_0 << v_{max}$ we have

$$\frac{dr}{dt} = -(1 - r(t)^2 A^2 e^{-2\beta\ t})$$

with $\beta = \frac{s^2}{1+s^2}$ and $A = sv_0$.

- The solution to the previous equation for $t_0 = 0$ and $r(0) = r_0$ is a complicated function $r(t; r_0)$

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- The light ray either diverges at finite t, either turns back at $r = \Lambda$ at finite t. The separating evolution of the light ray between the two behaviours is the position of the horizon with time t and is given by the request that the light ray diverges for infinite $t \to \infty$. This gives

$$r_{hor}(t) = \frac{e^{\beta t} I_0\left(\frac{Ae^{-\beta t}}{\beta}\right)}{A I_1\left(\frac{Ae^{-\beta t}}{\beta}\right)}.$$

- For large $t \to \infty$, the position of the horizon i given by

$$r_{hor}(t) \to \frac{2\beta}{A^2} e^{2\beta t} = \frac{2v_{max}^2}{v_0^2} e^{2\beta t}.$$

Hall conductivity

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- To study the hall conductivity we must assume a small electric field E_x in the x-direction and a large magnetic field B_z in the z-direction.

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- To study the hall conductivity we must assume a small electric field E_x in the x-direction and a large magnetic field B_z in the z-direction.
- Pure AdS spacetime has Lorentz invariance under boosts and rotations in the x, y, z, t directions. The electric and magnetic field transform under boosts:

$$
\begin{aligned}
\vec{E}_{||} &= \vec{E}_{||} \quad , \quad \vec{E}_{\perp} = \gamma\left(\vec{E}_{\perp} + \vec{v} \times \vec{B}\right) \\
\vec{B}_{||} &= \vec{B}_{||} \quad , \quad \vec{B}_{\perp} = \gamma\left(\vec{B}_{\perp} - \vec{v} \times \vec{E}\right) .
\end{aligned}
$$

By doing a boost with velocity $v_y = -E_x / B_z$ we have in the boosted frame $\vec{E}' = \vec{0} \quad , \quad \vec{B}' = \hat{z}\sqrt{B_z^2 - E_x^2}$.

Hall conductivity

- Therefore in this frame we have only a constant magnetic field B'_z in the z-direction and the motion of the string will be a spiral towards a fixed point as we have seen already. Therefore for large times, the velocity of the particle will be that of the boosted frame $v_y = -E_x/B_z$. From this, we deduce that we have the Hall conductivity

$$\sigma_{xy} = \frac{j_y}{E_x} = -\frac{q}{B_z}.$$

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- The motion for the quark is a spiral, as the radiation emmited absorbs continuousky its energy. The motion of the particle for late times is exponentially damped.

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- The induced string metric has a horizon that is time dependent and its position moves exponentially fast at late times towards the center of AdS. The induced string metric is locally that of AdS_2 like in any other string motion in bulk AdS.

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- The induced string metric has a horizon that is time dependent and its position moves exponentially fast at late times towards the center of AdS. The induced string metric is locally that of AdS_2 like in any other string motion in bulk AdS.
- The Hall conductivity is the same with the classical case.