Hawking Radiation by Schwarzschild-Tangherlini

Thomas Pappas
Theoretical Physics Division
University of Ioannina

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Schwarzschild-Tangherlini

$$ds^2 = \left[1 - \left(\frac{r_H}{r}\right)^{n+1}\right] dt^2 - \left[1 - \left(\frac{r_H}{r}\right)^{n+1}\right]^{-1} dr^2 - r^2 d\Omega_{2+n}^2$$

The Hypersphere element is given by:

$$d\Omega_{2+n}^2 := \left(d\theta_n{}^2 + sin^2\theta_n \left(d\theta_{n-1}{}^2 + sin^2\theta_{n-1}(\ldots(d\theta^2 + sin^2\theta d\varphi^2)\ldots)\right)\right)$$

extra dimensions are interpreted as extra angles.

The radius of the B.H:

$$r_H = \frac{1}{\sqrt{\pi} M_*} \left(\frac{M_{BH}}{M_*}\right)^{\frac{1}{n+1}} \left[\frac{8\Gamma\left(\frac{n+3}{2}\right)}{n+2}\right]^{\frac{1}{n+1}}$$

Eq. of motion

The eq. of motion for a scalar field:

$$\frac{1}{\sqrt{|g|}}\nabla_{\mu}\left(\sqrt{|g|}g^{\mu\nu}\nabla_{\nu}\Phi\right)=0$$

On the brane (projection of S-T metric)

$$ds^2=[1-\left(\frac{r_H}{r}\right)^{n+1}]dt^2-[1-\left(\frac{r_H}{r}\right)^{n+1}]^{-1}dr^2-r^2d\Omega^2$$

The radial part of e.o.m

$$\frac{h}{r^2}\frac{d}{dr}\left[hr^2\frac{dR}{dr}\right]+\left[\omega^2-\frac{h}{r^2}l(l+1)\right]R=0$$

Regge – Wheeler Potential

Write the last eq. in a Schrodinger like form

$$\left(h\frac{d}{dr}\right)^2 u + \left[\omega^2 - \frac{hh'}{r} - \frac{h}{r^2}l(l+1)\right]u = 0$$

Regge – Wheeler potential on the brane:

$$V_{eff} = h\left[\frac{h'}{r} + \frac{l(l+1)}{r^2}\right]$$

On the bulk we find:

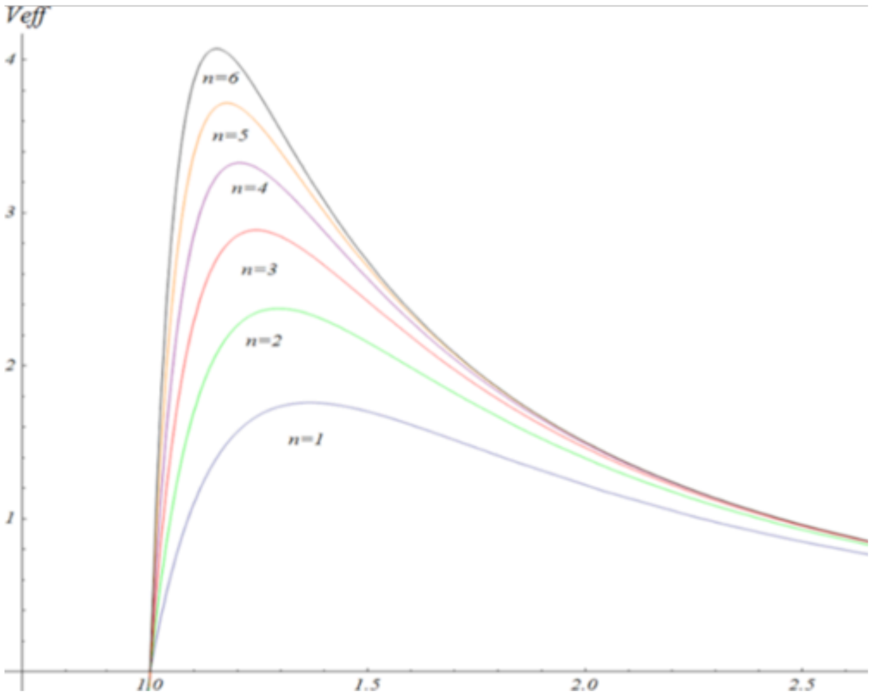
$$V_{eff}{}^{(Bulk)} = h[\frac{l(l+n+1)}{r^2} + \frac{h'}{r}\left(\frac{n+2}{2}\right) + \frac{n}{2}\left(\frac{n+2}{2}\right)\frac{h^2}{r^2}]$$

Depends on n & l. Located at aprox. 1.3 rh

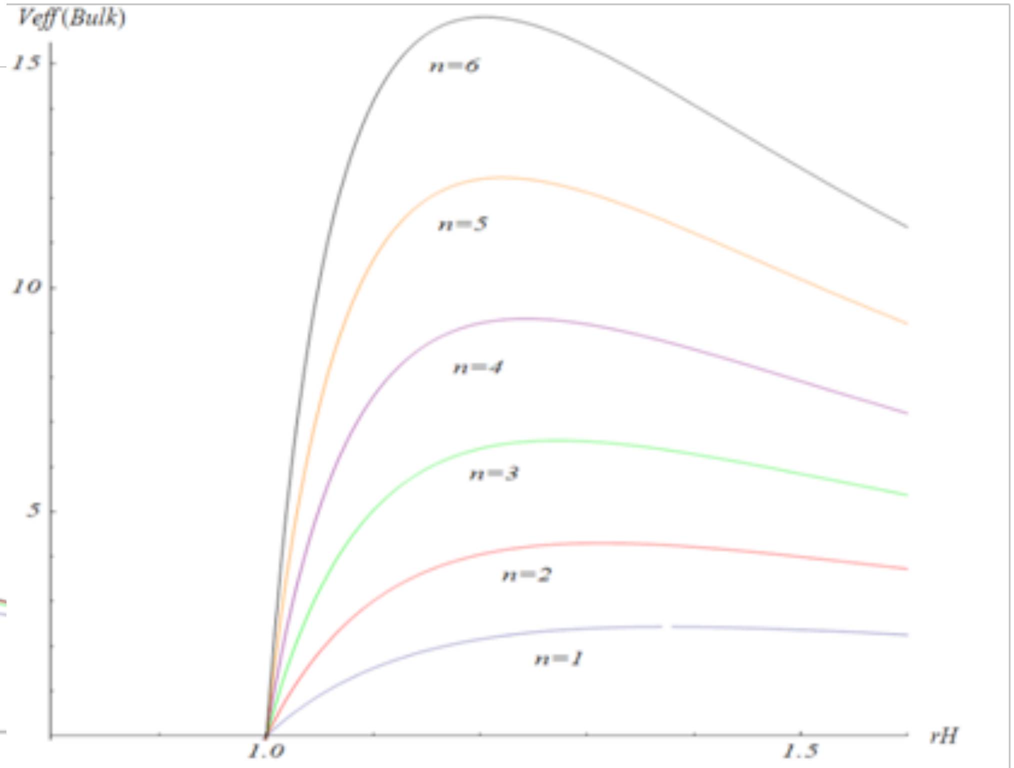
Responsible for non exact B.B radiation -> Backscattering effect

Regge – Wheeler Potential

Fixed mode of particles
(l=2)



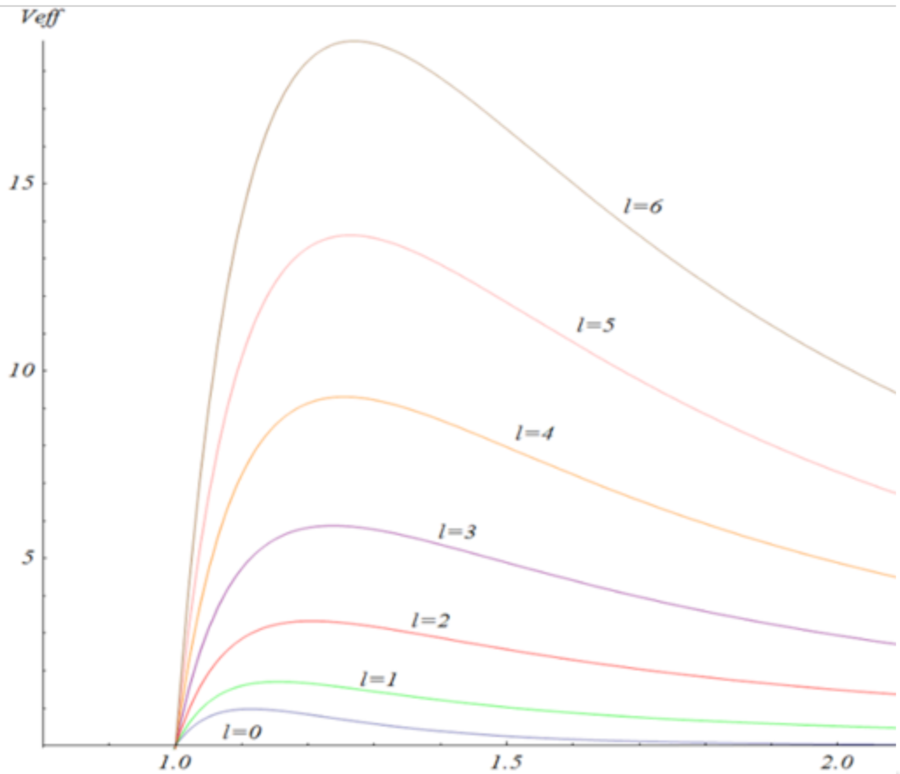
brane



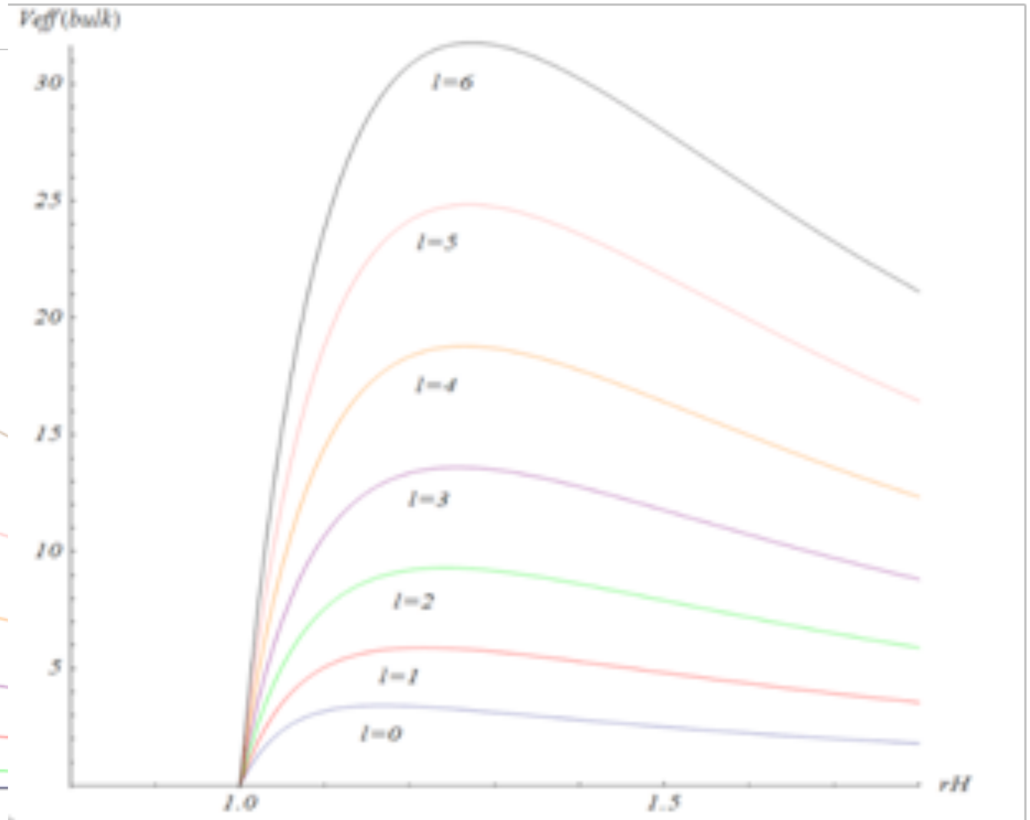
bulk

Regge – Wheeler Potential

Fixed # of extra dim
(n=4)



brane



bulk

Solving the equations of motion (brane)

$$\frac{h(r)}{r^2}\frac{d}{dr}\left[h(r)r^2\frac{d}{dr}\right]R(r)+\left[\omega^2-\frac{h(r)}{r^2}l(l+1)\right]R(r)=0$$

Where: $h(r)=1-\left(\frac{r_H}{r}\right)^{n+1}$ Not exactly solvable!

Near the Horizon Solution:

We take: $r\to h(r)$

$$h(1-h)\frac{d^2R}{dh^2}+\left[1-\frac{(2n+1)}{(n+1)}h\right]\frac{dR}{dh}$$

$$+\left[\frac{(\omega r_H)^2}{(n+1)^2h(1-h)}-\frac{l(l+1)}{(n+1)^2(1-h)}\right]R=0$$

Solving the equations of motion (brane)

We define a new radial function: $R(h) = h^{\alpha}(1-h)^{\beta}F(h)$

$$h(1-h)\frac{d^2F}{dh^2} + \left[2\alpha + 1 - 2\alpha h - 2\beta h - \frac{n}{(n+1)}h - h\right]\frac{dF}{dh}$$
$$+ \left[\alpha(\alpha-1)\frac{(1-h)}{h} - 2\alpha\beta + \beta(\beta-1)\frac{h}{1-h} + \frac{\alpha}{h}\right.$$
$$- \frac{\beta}{1-h} - \frac{2n+1}{n+1}\left(\alpha - \beta\frac{h}{1-h}\right)$$
$$\left. + \frac{1}{(n+1)^2}\left[\frac{(\omega r_H)^2}{h(1-h)} - \frac{l(l+1)}{(1-h)}\right]\right]F = 0$$

Hypergeometric diff. eq. with general solution…

Solving the equations of motion (brane)

$$R_{NH}(h) = A_{-}\left(\frac{r_H}{r}\right)^{\beta(n+1)} \frac{\Gamma(2\alpha+1)\Gamma\left(1-2\beta-\frac{n}{n+1}\right)}{\Gamma\left(1+\alpha-\beta-\frac{n}{n+1}\right)\Gamma(1+\alpha-\beta)}$$

$$+ A_{-}(1-h)^{1-\beta-\frac{n}{n+1}} \frac{\Gamma(2\alpha+1)\Gamma\left(2\beta+\frac{n}{n+1}-1\right)}{\Gamma\left(\alpha+\beta+\frac{n}{n+1}\right)\Gamma(\alpha+\beta)}$$

Where:

$$\alpha = -\frac{i\omega r_H}{n+1}$$

$$\beta = \frac{1}{2(n+1)}[1-\sqrt{(2l+1)^2-4(\omega r_H)^2}]$$

For low energy particles we have: $\omega r_H \ll 1 \Longrightarrow \beta \approx \frac{-l}{n+1}$

Solving the equations of motion (brane)

$$R_{NH}(h) = A_-\left(\frac{r}{r_H}\right)^l \frac{\Gamma(2\alpha+1)\Gamma\left(1-2\beta-\frac{n}{n+1}\right)}{\Gamma\left(1+\alpha-\beta-\frac{n}{n+1}\right)\Gamma(1+\alpha-\beta)}$$
$$+ A_-\left(\frac{r_H}{r}\right)^{l+1} \frac{\Gamma(2\alpha+1)\Gamma\left(2\beta+\frac{n}{n+1}-1\right)}{\Gamma\left(\alpha+\beta+\frac{n}{n+1}\right)\Gamma(\alpha+\beta)}$$

A

Far away from the Horizon:

$$R(r) = f(r)r^{-1/2} \qquad z \coloneqq \omega r$$

$$\frac{d^2f}{dz^2} + \frac{1}{z}\frac{df}{dz} + \left[1 - \frac{\left(l+\frac{1}{2}\right)^2}{z^2}\right]f = 0$$

Solving the equations of motion (brane)

$$R_{FF}(r) = \left(\frac{\omega}{2}\right)^{l+\frac{1}{2}} r^l \frac{B_+}{\Gamma(l+\frac{3}{2})} - \frac{B_-}{\pi}\frac{\Gamma\left(l+\frac{1}{2}\right)}{r^{l+1}}\left(\frac{2}{\omega}\right)^{l+\frac{1}{2}}$$

B

Matching of the solutions:

$$\frac{B_+}{B_-} = -\left(\frac{2}{\omega r_H}\right)^{2l+1} \frac{\Gamma\left(l+\frac{1}{2}\right)^2\left(l+\frac{1}{2}\right)\Gamma\left(1-2\beta-\frac{n}{n+1}\right)\Gamma(\alpha+\beta)\Gamma\left(\alpha+\beta+\frac{n}{n+1}\right)}{\pi\Gamma\left(1+\alpha-\beta-\frac{n}{n+1}\right)\Gamma(1+\alpha-\beta)\Gamma\left(2\beta+\frac{n}{n+1}-1\right)}$$

$$R_{FF}(r) \approx \frac{B_+ - iB_-}{\sqrt{2\omega\pi r^{n+2}}}\exp\left[i\left(\omega r - \frac{n+2}{4}\pi\right)\right] + \frac{B_+ + iB_-}{\sqrt{2\omega\pi r^{n+2}}}\exp\left[-i\left(\omega r - \frac{n+2}{4}\pi\right)\right]$$

Spherical waves at infinity.

Solving the equations of motion (brane & bulk)

$$|A|^2 \approx \frac{16\pi}{(n+1)^2}\left(\frac{\omega r_H}{2}\right)^{2l+2} \frac{\Gamma\left(\frac{l+1}{n+1}\right)^2 \Gamma\left(1+\frac{l}{n+1}\right)^2}{\Gamma\left(l+\frac{1}{2}\right)^2 \Gamma\left(1+\frac{2l+1}{n+1}\right)^2}$$

The gray body factor is given by:

$$\sigma(\omega) = \frac{2^n \pi^{\frac{n+1}{2}}}{n!\,\omega^{n+2}} \Gamma\left[\frac{n+1}{2}\right] \frac{(2l+n+1)(l+n)!}{l!} |A|^2$$

For the l=0 mode the gray body factor equals the surface of the B.H for every n.

Solving the equations of motion (bulk)

$$\frac{h}{r^{n+2}}\frac{d}{dr}\left[r^{n+2}h(r)\frac{d}{dr}\right]R(r)+[\omega^2-\frac{h}{r^2}l(l+n+1)]R(r)=0$$

Near the Horizon Solution:

$$R_{NH}(h)\approx A_-\left[\left(\frac{r}{r_H}\right)^l\frac{\Gamma(2\alpha+1)\Gamma(1-2\beta)}{\Gamma(1+\alpha-\beta)^2}+\left(\frac{r_H}{r}\right)^{l+n+1}\frac{\Gamma(2\alpha+1)\Gamma(2\beta-1)}{\Gamma(\alpha+\beta)^2}\right]$$

Solution far away from Horizon:

$$R_{FF}(r)=\frac{B_+r^l}{\Gamma\left(l+\frac{n+3}{2}\right)}\left(\frac{\omega}{2}\right)^{l+\frac{n+1}{2}}-\frac{B_-}{r^{l+n+1}}\frac{\Gamma\left(l+\frac{n+1}{2}\right)}{\pi}\left(\frac{2}{\omega}\right)^{l+\frac{n+1}{2}}$$

Solving the equations of motion (bulk)

Matching of the solutions:

$$|A|^2 \approx \frac{4\pi^2}{2^{\frac{4l}{n+1}}}\left(\frac{\omega r_H}{2}\right)^{2l+n+2} \frac{\Gamma\left(1+\frac{l}{n+1}\right)^2}{\Gamma\left(\frac{1}{2}+\frac{l}{n+1}\right)^2 \Gamma\left(l+\frac{n+3}{2}\right)^2}$$

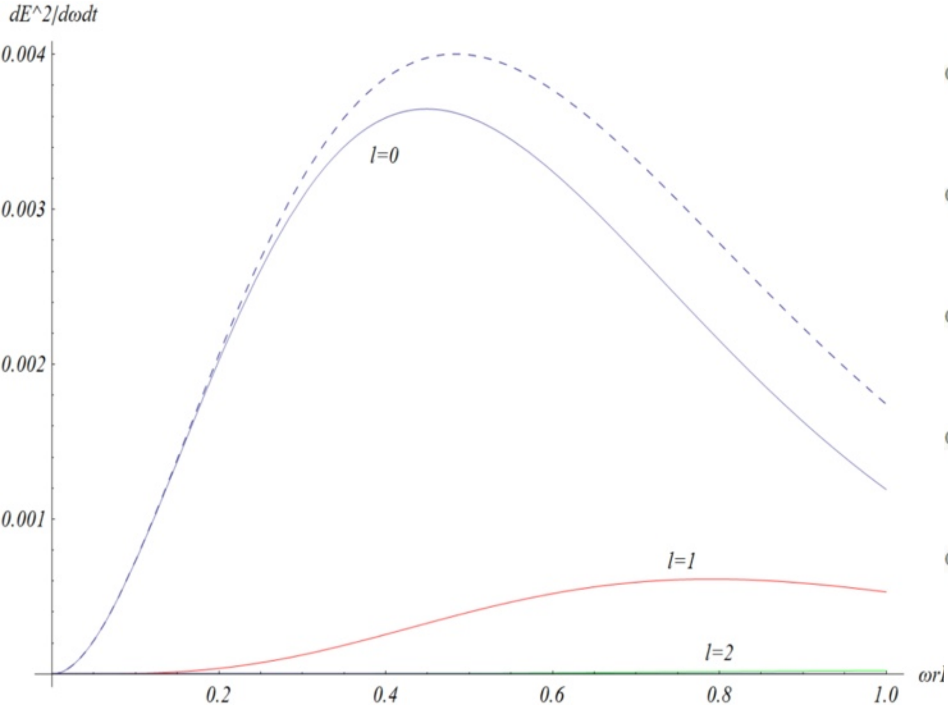
The energy emission rate is given by:

$$\frac{dE}{dt d\omega} = \frac{1}{2\pi}\sum_j N_j \left|A_j\right|^2 \frac{\omega}{\exp\left[\frac{\omega}{T_{BH}}\right]-1}$$

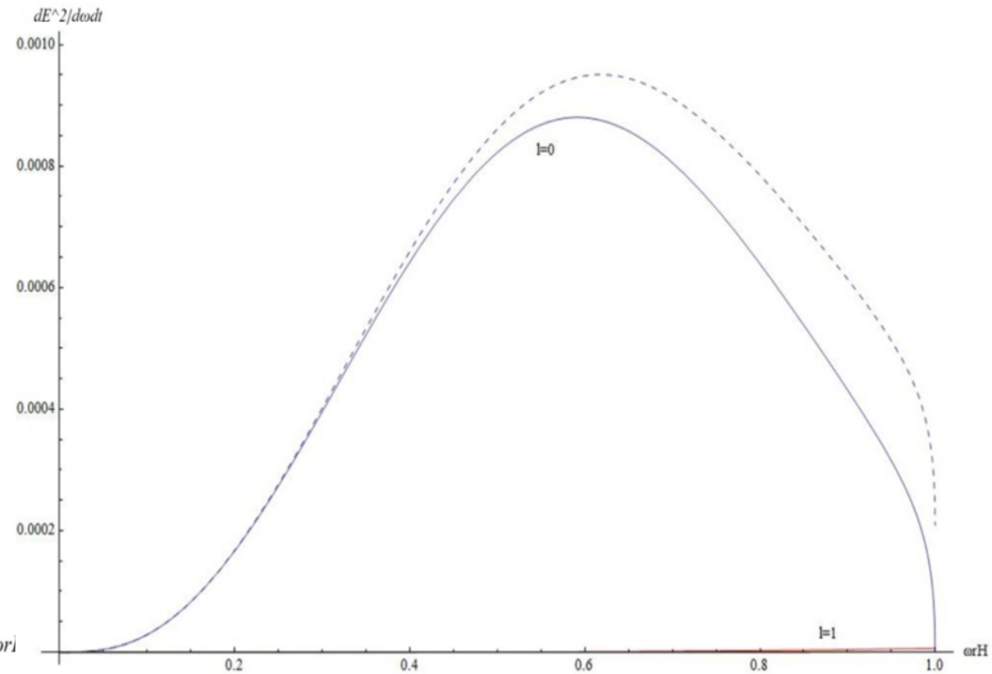
With: $$N_j := \frac{(2j+n+1)(j+n)!}{j!\,(n+1)!}$$

Energy Emission Rates

(Fixed # of extra dim.
n=1)



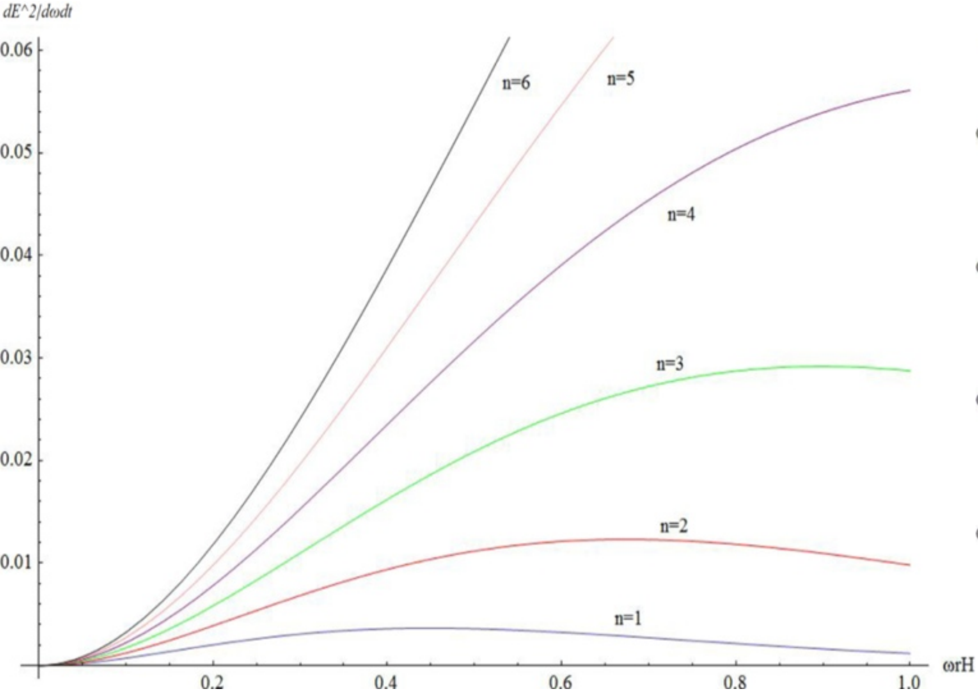
brane

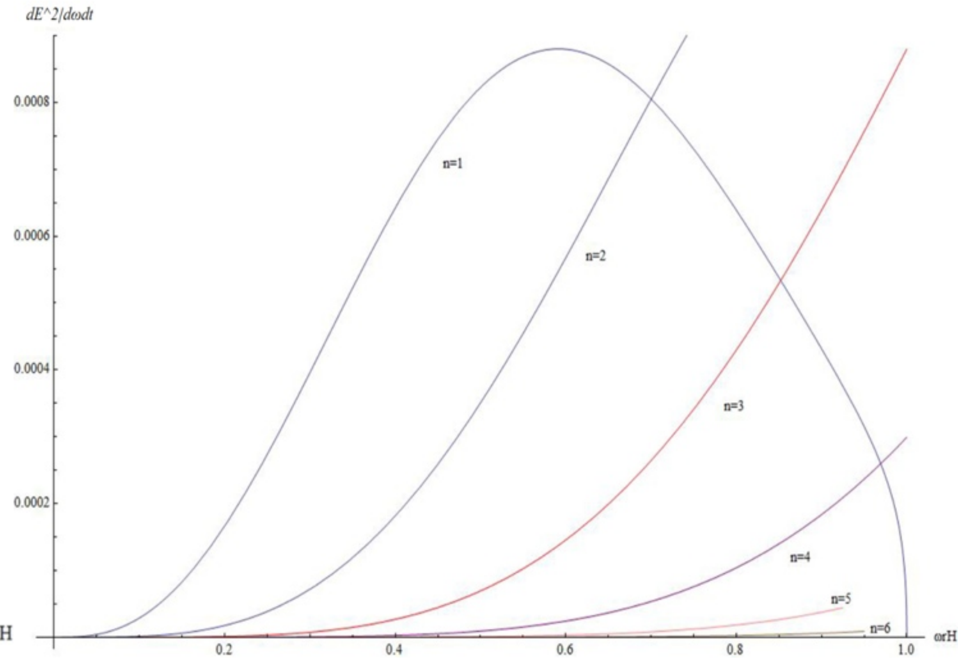


bulk

Energy Emission Rates

(For the dominant mode
l=0)



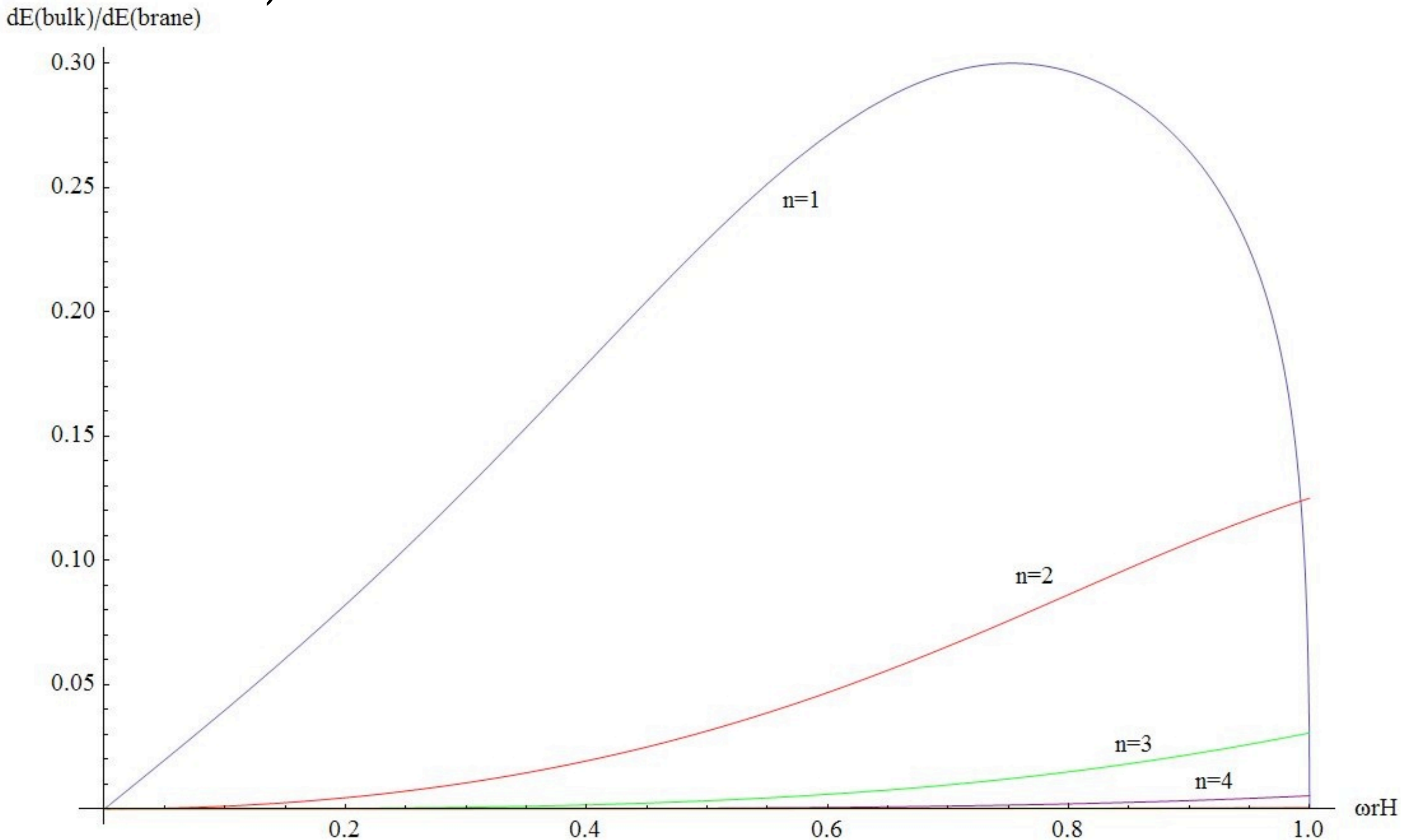


brane

bulk

Energy Emission Rates

(Ratio of bulk over brane energy emission for the dominant mode l=0)



That’s all Folks!