Supersymmetric Localization

Bruno Le Floch[1,2]

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[1] École Normale Supérieure, Paris, France
[2] Perimeter Institute, Waterloo, Canada

Goal: compute the vev of SUSYc operators in (Euclidean) SUSYc theories,

$$\langle \mathcal{O} \rangle = \int [D\varphi]\, \mathcal{O}[\varphi]\, e^{-S_E[\varphi]}\,.$$

Step 1: deform action $S_E \to S_E + t\{Q, V\}$ with Q a supercharge, V a (nice) combination of fields. VEVs do not depend on $t \in \mathbb{R}$:

$$\begin{aligned}
\frac{\partial}{\partial t} \langle \mathcal{O} \rangle_t &= \frac{\partial}{\partial t} \int [D\varphi]\, \mathcal{O}\, e^{-S_E - t\{Q,V\}} \\
&= - \int [D\varphi]\, \{Q, V\}\, \mathcal{O}\, e^{-S_E - t\{Q,V\}} \\
&= - \int [D\varphi] \left\{ Q, V\, \mathcal{O}\, e^{-S_E - t\{Q,V\}} \right\} = 0\,,
\end{aligned}$$

if $[Q, \mathcal{O}] = [Q, S_E] = [Q, \{Q, V\}] = 0$ and $[D\varphi]$ is invariant.

As $t \to \infty$ the saddle point approximation $\varphi = \varphi_0 + \frac{1}{\sqrt{t}}\delta\varphi$ around saddle points of $\{Q, V\}$ becomes exact (if $\{Q, V\} \geq 0$):

$$\begin{aligned}\langle \mathcal{O} \rangle_{t=0} &= \lim_{t\to\infty} \langle \mathcal{O} \rangle_t = \lim_{t\to\infty} \int [D\varphi]\, \mathcal{O}\, e^{-S_E - t\{Q,V\}} \\ &= \int_{\text{saddle}} [D\varphi_0] \left(\mathcal{O}[\varphi_0]\, e^{-S_E[\varphi_0] - t\{Q,V[\varphi_0]\}} \right. \\ &\qquad \left. \cdot \int [D\delta\varphi]\, e^{-\frac{1}{2}\partial_\varphi^2\{Q,V[\varphi_0]\}(\delta\varphi)^2} \right) \\ &= \int_{\substack{\text{saddle} \\ \{Q,V\}=0}} [D\varphi_0]\, \mathcal{O}_{\text{cl}}\, Z_{\text{cl}}\, Z_{\text{one-loop}}\,.\end{aligned}$$

Step 2: find saddle points of $\{Q, V\}$ with $\{Q, V\} = 0$.
Step 3: compute $Z_{\text{one-loop}}$.
Step 4: profit. $\langle \mathcal{O} \rangle$ is then known whenever $[Q, \mathcal{O}] = 0$.

Choice of Q and $V \implies$ lots of freedom! Constraints:

- Q is a symmetry,
- $[Q^2, V] = 0$.

This can give distinct expressions for a single VEV. For example, S^2 partition function of $\mathcal{N} = (2,2)$ SQED:

$$Z = \sum_B \int \mathrm{d}a\, e^{-4\pi\xi a + i\vartheta B} \prod_{s=1}^{N_f} \frac{\Gamma(-iM_s - ia - B/2)}{\Gamma(1 + iM_s + ia - B/2)} \prod_{s=1}^{\widetilde{N}_f} \frac{\Gamma(-i\widetilde{M}_s - ia - B/2)}{\Gamma(1 + i\widetilde{M}_s + ia - B/2)}$$

$$= \sum_{\text{discrete } a} e^{-4\pi\xi a} Z_{\text{one-loop}} Z_V \widetilde{Z_V}$$

$$= \text{Toda correlator}\ldots$$

Thanks!