Gerbes and Generalized Kähler Manifold

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S^1 bundle

Our starting point of the following construction will be a closed two-form ω.

$$\omega = \mathrm{d}A_\alpha$$
$$\delta(\mathrm{d}A)_{\alpha\,\beta} = \mathrm{d}(\delta A_{\alpha\,\beta}) = \omega_\alpha - \omega_\beta = 0$$
$$\Rightarrow \delta A_{\alpha\,\beta} = A_\alpha - A_\beta = \mathrm{d}\Lambda_{\alpha\,\beta}$$
$$\delta(\mathrm{d}\Lambda)_{\alpha\,\beta\,\gamma} = \mathrm{d}(\delta\Lambda_{\alpha\,\beta\,\gamma}) = \delta(\delta A)_{\alpha\,\beta\,\gamma} = 0$$
$$\Rightarrow \delta\Lambda_{\alpha\,\beta\,\gamma} = \Lambda_{\alpha\,\beta} - \Lambda_{\alpha\,\gamma} + \Lambda_{\beta\,\gamma} = d_{\alpha\,\beta\,\gamma}.$$

Then we can define new functions $g_{\alpha\,\beta} = e^{i\Lambda_{\alpha\,\beta}}$ which satisfy following relations

$$g_{\alpha\,\beta} : U_\alpha \cap U_\beta \to S^1$$
$$g_{\alpha\,\beta} = g^{-1}_{\beta\,\alpha}$$
$$g_{\alpha\,\beta}\ g_{\beta\,\gamma}\ g_{\gamma\,\alpha} = 1$$

By analogy with the previous approach we start with $\frac{H}{2\pi} \in H^3(M, \mathbb{Z})$. The Poincaré lemma implies the following relations

$$H = \mathrm{d}B_\alpha$$
$$\delta B_{\alpha\,\beta} = B_\alpha - B_\beta = \mathrm{d}A_{\alpha\,\beta}$$
$$\delta A_{\alpha\,\beta\,\gamma} = A_{\alpha\,\beta} - A_{\alpha\,\gamma} + A_{\beta\,\gamma} = \mathrm{d}\Lambda_{\alpha\,\beta\,\gamma}$$
$$\delta\Lambda_{\alpha\,\beta\,\gamma\,\phi} = \Lambda_{\alpha\,\beta\,\gamma} - \Lambda_{\alpha\,\beta\,\phi} + \Lambda_{\alpha\,\gamma\,\phi} - \Lambda_{\beta\,\gamma\,\phi} = d_{\alpha\,\beta\,\gamma\,\phi} \in 2\pi\mathbb{Z}$$

The set of functions $g_{\alpha\,\beta\,\gamma} = e^{i\Lambda_{\alpha\,\beta\,\gamma}}$ satisfying the following chain of relations

$$g_{\alpha\,\beta\,\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$$
$$g_{\alpha\,\beta\,\gamma} = g_{\gamma\,\alpha\,\beta} = g_{\beta\,\gamma\,\alpha} = g^{-1}_{\beta\,\alpha\,\gamma} = g^{-1}_{\alpha\,\gamma\,\beta} = g^{-1}_{\gamma\,\beta\,\alpha}$$
$$g_{\alpha\,\beta\,\gamma}\; g_{\alpha\,\beta\,\phi}\; g_{\alpha\,\gamma\,\phi}\; g_{\beta\,\gamma\,\phi} = 1$$

are transition functions of a gerbe with connection.

$(M, J^{\pm}, g, H)$ is a generalized Kähler manifold if the following axioms hold

$$
\begin{aligned}
(J^{\pm})^2 &= -\mathbb{I} \\
N^{\pm}(X, Y) &= 0 \\
J^{\pm\, t} g J^{\pm} &= g \\
\nabla^{\pm} J &= 0 \\
\Gamma^{\pm}{}_{j\,k}{}^{i} &= \Gamma_{j\,k}^{\,i} \pm \frac{1}{2} H_{j\,k}^{\,i} \\
\mathrm{d}H &= 0.
\end{aligned}
$$

A generalized Kähler manifold is locally encoded by a real potential $K(z, \bar{z}, w, \bar{w}, X_L, \bar{X}_L, X_R, \bar{X}_R)$

$$K_\alpha - K_\beta = F^+_{\alpha\,\beta}(z, w, X_L) + \bar{F}^+_{\alpha\,\beta}(\bar{z}, \bar{w}, \bar{X}_L) + F^-_{\alpha\,\beta}(z, \bar{w}, X_R) + \bar{F}^-_{\alpha\,\beta}(\bar{z}, w, \bar{X}_R)$$

$$F^+_{\alpha\,\beta}(z, w, X_L) + F^+_{\alpha\,\gamma}(z, w, X_L) + F^+_{\beta\,\gamma}(z, w, X_L) = i\big(c_{\alpha\,\beta\,\gamma}(z) - b_{\alpha\,\beta\,\gamma}(w)\big)$$
$$\bar{F}^+_{\alpha\,\beta}(\bar{z}, \bar{w}, \bar{X}_L) + \bar{F}^+_{\alpha\,\gamma}(\bar{z}, \bar{w}, \bar{X}_L) + \bar{F}^+_{\beta\,\gamma}(\bar{z}, \bar{w}, \bar{X}_L) = -i\big(\bar{c}_{\alpha\,\beta\,\gamma}(\bar{z}) - \bar{b}_{\alpha\,\beta\,\gamma}(\bar{w})\big)$$
$$F^-_{\alpha\,\beta}(z, \bar{w}, X_R) + F^-_{\alpha\,\gamma}(z, \bar{w}, X_R) + F^-_{\beta\,\gamma}(z, \bar{w}, X_R) = -i\big(c_{\alpha\,\beta\,\gamma}(z) + \bar{b}_{\alpha\,\beta\,\gamma}(\bar{w})\big)$$
$$\bar{F}^-_{\alpha\,\beta}(\bar{z}, w, \bar{X}_R) + \bar{F}^-_{\alpha\,\gamma}(\bar{z}, w, \bar{X}_R) + \bar{F}^-_{\beta\,\gamma}(\bar{z}, w, \bar{X}_R) = i\big(\bar{c}_{\alpha\,\beta\,\gamma}(\bar{z}) + b_{\alpha\,\beta\,\gamma}(w)\big)$$

$$c_{\alpha\,\beta\,\gamma} + c_{\alpha\,\beta\,\phi} + c_{\alpha\,\gamma\,\phi} + c_{\beta\,\gamma\,\phi} = \frac{i}{4} d_{\alpha\,\beta\,\gamma\,\phi}$$
$$b_{\alpha\,\beta\,\gamma} + b_{\alpha\,\beta\,\phi} + b_{\alpha\,\gamma\,\phi} + b_{\beta\,\gamma\,\phi} = \frac{i}{4} d_{\alpha\,\beta\,\gamma\,\phi},$$

We can introduce a biholomorphic gerbe $G_{\alpha\,\beta\,\gamma}(z)$ and a twisted biholomorphic gerbe $F_{\alpha\,\beta\,\gamma}(w)$

$$G_{\alpha\,\beta\,\gamma}(z) = e^{4 c_{\alpha\,\beta\,\gamma}(z)}$$
$$F_{\alpha\,\beta\,\gamma}(w) = e^{4 b_{\alpha\,\beta\,\gamma}(w)}$$