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# PHASE TRANSITIONS IN ABELIAN LATTICE GAUGE THEORIES

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We study the Euclidean partition function of Abelian lattice (gauge) theories in various dimensions. Using generalizations of mathematical methods developed recently to study the XY model in two dimensions, we obtain useful expressions for the partition functions and physical pictures of the phases of these more complicated theories. Approximate duality relations and dilute gas approximations yield estimates of critical coupling constants which separate confining and non-confining phases for the rotor model in three dimensions and Abelian lattice gauge theory in four dimensions. Generalizations of this work to non-Abelian continuum theories are discussed.

## 1. Introduction

The strong coupling approach to lattice gauge theory [1] promises to provide us with a systematic and convergent approximation to the masses and static properties of the low-lying hadrons in QCD. The basic premise of this approach is that the theory has no "phase transition" which would prevent us from extrapolating from the strong to the weak coupling region. This assumption is tested automatically in each order of the strong coupling Padé approximants and to lowest nontrivial order the results are attractive: Abelian gauge theories have a transition in 4 dimensions while non-Abelian ones do not.

However, even if this attractive state of affairs survives higher order calculations, we will be uneasy about the fact that Padé approximants understand the difference

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between Abelian and non-Abelian gauge theories better than we do. In this paper we will attempt to remove this feeling of uneasiness by presenting an intuitive physical picture of the phase transitions in Abelian lattice theories.

The crucial ingredients in this picture are certain solutions of the continuum classical field equations with singular sources. Polyakov [2] has pioneered the investigation of phase transitions in terms of Euclidean classical fields, and many of our results were anticipated by him. However we obtain these results by a series of exact transformations without appealing to the method of steepest descent. Thus our methods are not necessarily restricted to the regime of extremely weak coupling and we can investigate phase transitions at finite $g$.

The method we shall use is a direct generalization of the technique used by José, Kadanoff, Kirkpatrick and Nelson (JKKN) [3] to derive the Kosterlitz-Thouless [4] "vortex + spinwave" picture of the two-dimensional plane rotor model. We will find an analogous "monopole + photon" picture of lattice gauge theory in 3 dimensions and a "monopole current loop + photon" picture in 4 dimensions. In 5 dimensions we would have monopole surface charges etc. The method also generalizes to plane rotor models in higher dimensions, and we find an intimate connection between the $d$-dimensional rotor model and the $d + 1$ dimensional gauge theory.

Unfortunately, we do not at present know how to extend the JKKN method to theories with non-Abelian symmetry groups. We do believe that the intuition we have gained from the Abelian theories will be helpful in understanding the non-Abelian case as well. This will be discussed in the last section of the paper.

After this work was completed we received interesting preprints by Savit [5] and Glimm and Jaffe [6] dealing with Abelian lattice gauge theories. These authors have arrived at the picture of the low temperature phase of these theories that is presented in this paper. We expect that many of the results we present here are known to the above authors as well as to other workers actively engaged in this field.

## 2. Three-dimensional gauge model

We begin by studying the three-dimensional Abelian lattice gauge theory. The most important correlation function in this theory is Wilson's [1] line integral, $\langle\exp(i \int A_\mu \, dx_\mu)\rangle$ which is given by $e^{-W(C)} = Z(J)/Z(0)$ with

$$Z(J) \equiv \prod_{r,\mu} \int_0^{2\pi} d\theta_\mu(r) \exp[-\beta\Sigma_{r,\mu,\nu} (1 - \cos\theta_{\mu\nu}(r))] \exp[i\Sigma_{r,\mu}\theta_\mu(r) J_\mu(r)] \,. \tag{1}$$

$\theta_\mu(r)$ is the angle on the directed link $r \to r + \hat{\mu}$ and

$$\theta_{\mu\nu}(r) = \Delta_\mu\theta_\nu(r) - \Delta_\nu\theta_\mu(r) \tag{2}$$

with $\Delta_\mu$ a finite difference operator. The current $J_\mu$ is defined by

$$J_\mu(r) = \begin{cases} +1 \text{ if link } r \to r + \hat{\mu} \text{ is in the contour C} \\ -1 \text{ if link } r + \hat{\mu} \to r \text{ is in the contour C} \\ 0 \text{ otherwise.} \end{cases} \tag{3}$$

The integrand of (1) is a periodic function of each $\theta_{\mu\nu}(r)$ variable and so can be expanded in a Fourier series

$$e^{-\beta(1-\cos\theta_{\mu\nu})} = \sum_{l_{\mu\nu}=-\infty}^{\infty} e^{il_{\mu\nu}\theta_{\mu\nu}}\, e^{-\beta} I_{l_{\mu\nu}}(\beta) \,. \tag{4}$$

For large $\beta$ the modified Bessel function in (4) can be approximated by

$$e^{-\beta} I_l(\beta) \underset{\beta\to\infty}{\sim} e^{-l^2/4\beta}/\sqrt{2\pi\beta} \,. \tag{5}$$

We will take the Fourier transform of the Gibbs factor to be given by (5) for all $\beta$. This defines the Villain [7] or periodic Gaussian model. We would certainly expect this model to be in the same universality class as the original interaction and thus to have the same kind of phase diagram. Furthermore, as far as continuum gauge theories are concerned, the two models are on the same footing. They both provide gauge invariant regularizations of the continuum theory which exhibit confinement in the strong coupling limit.

Using the expansion (4) we can do all of the $\theta_\mu$ integrals. Each $\theta_\mu$ couples to 4 $l_{\mu\nu}$'s and we obtain

$$Z(J) = \sum_{r,\mu,\nu\;\, l_{\mu\nu}(r)=-\infty} \prod_{r,\mu} \delta_{\Delta_\nu l_{\mu\nu}(r)+J_\mu(r),0} \exp\left[-\frac{1}{4\beta}\sum l_{\mu\nu}^2\right]. \tag{6}$$

The sum over $l_{\mu\nu}$ in (6) is constrained, and the simplest way to proceed is to solve the constraint equation. The general solution is

$$l_{\mu\nu}(r) = n^\mu(n\cdot\Delta)^{-1} J^\nu - n^\nu (n\cdot\Delta)^{-1} J^\mu + \epsilon_{\mu\nu\lambda}\, \Delta_\lambda l(r) \,. \tag{7}$$

Here $l(r)$ is an integer valued scalar field determined up to a constant by $J_\mu$ and $l_{\mu\nu}$ (The constant is fixed by boundary conditions at $\infty$.) and $n_\lambda$ is a unit vector in the 3 direction. We have also used the notation

$$[(n\cdot\Delta)^{-1} f](r) = \sum_{m=-\infty}^{r_3} f(r_1, r_2, m) \,.$$

The constrained sum over $l_{\mu\nu}$ now becomes an unconstrained sum over $l$ which however converges rather slowly for large $\beta$. We improve this convergence by using the Poisson sum formula: i.e. the Fourier series representation of the delta function,

$$\sum_l \delta(x-l) = \sum_m e^{2\pi i m x}$$

Equivalently,

$$\sum_{l=-\infty}^{\infty} g(l) = \sum_{m=-\infty}^{\infty} \int d\phi\, g(\phi)\, e^{2\pi i m\phi}\,,$$

where $g$ is an arbitrary function. Now eq. (6) becomes

$$Z(J) = \prod_r \int_{-\infty}^{\infty} d\varphi(r) \sum_{m(r)=-\infty}^{\infty} \left\{ \exp\left[\frac{-1}{2\beta}\sum_{r,\lambda} (\Delta_\lambda \varphi)^2\right] \exp\left[\frac{1}{\beta}\sum_{r,ij} \epsilon_{ij}(n\cdot\Delta)^{-1} J_i \Delta_j \varphi\right] \right.$$

$$\left. \times \exp\left[2\pi i \sum_r m(r)\,\varphi(r)\right] \right\} \exp\left[-\frac{1}{2\beta}\sum_{r,i} [(n\cdot\Delta)^{-1} J_i]^2\right] \tag{8}$$

(latin indices run over 1,2).

Finally, we do the Gaussian functional integral over $\varphi$, and obtain

$$e^{-W(C)} = \exp\left[-\frac{1}{2\beta}\sum_{r,r'\mu} J_\mu(r)\, v(r-r')\, J_\mu(r')\right]$$

$$\times \frac{\sum\limits_{m(r)=-\infty}^{\infty} \exp[-2\pi\beta\Sigma m(r)\, v(r-r')\, m(r')] \exp[2\pi i \Sigma \epsilon_{\mu\nu\lambda} n_\mu (n\cdot\Delta)^{-1} \Delta_\nu J_\lambda(r)\, v(r\quad r')\, m(r')]}{\sum\limits_{m(r)=-\infty}^{\infty} \exp[-2\pi\beta\Sigma m(r)\, v(r-r')\, m(r')]}\,, \tag{9}$$

where $\Delta_\nu^2\, v(r-r') = -\delta_{rr'}$, i.e. $v(r)$ is the lattice Coulomb Green function.

Eq. (9) shows us that our problem is equivalent to a gas of magnetic monopoles interacting with a stationary electric current loop. The identification of $m(r)$ with the magnetic monopole density is clear from eq. (9): $m(r)$ couples via $v(r-r')$ to $\Delta^\nu B_\nu(r')$ where $B_\nu(r')$ is the magnetic field generated by $J_\mu$,

$$B_\nu(r) \equiv \epsilon_{\nu\lambda\mu}\, n_\mu\, (n\cdot\Delta)^{-1} J_\lambda(r)\,.$$

Direct substition verifies that $B_\nu(r)$ satisfies the three-dimensional Euclidean Maxwell equation

$$\epsilon^{\sigma\rho\nu} \Delta_\rho B_\nu(r) = J^\sigma(r)\,,$$

so $B_\nu(r)$ is indeed the relevant magnetic field.

A familiar argument [8] shows that $e^{-W(C)}$ does not depend on which coordinate axis we choose for the direction of $n_\mu$, as long as $J_\mu$ and $m$ are integers.

The 3-dimensional Coulomb gas of eq. (9) is always in a plasma phase. The only thing that changes as we vary $\beta$ is the density of monopoles. (We are allowing monopoles to be created or destroyed.) We can then take over Polyakov's [2] arguments to show that for arbitrarily large finite $\beta$ we have a mass gap and the correlation function $e^{-W(C)}$ falls like $e^{-\text{area}(C)}$ for large area. Thus three-dimensional Abelian lattice gauge theory has no phase transition and the potential between static charges rises linearly with the separation. A lattice version of these arguments appears in appendix A.

As the lattice spacing and $1/\beta$ go zero (the usual continuum limit) the Coulomb self-energy of the monopoles blows up and their density goes to zero exponentially. This causes the coefficient of the linear force law to go to zero and we regain the usual continuum theory – free electromagnetism in three dimensions.

The informed reader will have noted that our derivation followed precisely the JKKN [3] argument for the plane rotor models. The results are also precisely the same. After duality transformation the two theories become dimensional continuations of a single model, the classical Coulomb gas, on a lattice.

## 3. Three-dimensional O(2) classical Heisenberg model

The basic variable here is a two-component unit vector $s = \binom{\cos\theta}{\sin\theta}$. The spin correlation function is given by

$$\langle S_+(R)\, S_-(0)\rangle \equiv \langle S_1 + iS_2(R), S_1 - iS_2(0)\rangle$$

$$= \frac{\prod_r \int_0^{2\pi} \mathrm{d}\theta(r) \exp[-\beta \sum_{r,\mu} (1-\cos\Delta_\mu\theta)] \exp\{i[\theta(R)-\theta(0)]\}}{\prod_r \int_0^{2\pi} \mathrm{d}\theta(r) \exp[-\beta \sum_{r,\mu}(1-\cos\Delta_\mu\theta)]} . \tag{10}$$

Again we Fourier transform and make the Villain approximation

$$\langle S_+(R)\, S_-(0)\rangle = \frac{\prod_{r,\mu} \sum_{l_\mu(r)=-\infty} \prod_r \delta_{\Delta_\mu l_\mu(r), Q(r)} \exp\left(-\frac{1}{4\beta}\Sigma l_\mu^2\right)}{\prod_{r,\mu} \sum_{l_\mu(r)=-\infty} \prod_r \delta_{\Delta_\mu l_\mu(r),0} \exp\left(-\frac{1}{4\beta}\Sigma l_\mu^2\right)} . \tag{11}$$

Here $Q(r) \equiv \delta_{r,R} - \delta_{r,0}$.

The constraint equation is solved by

$$l_\mu = n_\mu(n\cdot\Delta)^{-1}\, Q(r) + \epsilon_{\mu\nu\lambda}\, \Delta_\nu S_\lambda . \tag{12}$$

In this case however $S_\lambda$ is not determined uniquely by $l_\mu$ and $Q$: we have to choose a gauge. It is important that $S_\lambda$ be an integer whenever $l_\mu$ and $Q$ are and for this reason it is convenient to choose the axial gauge $n \cdot S = 0$. With this choice we have an unconstrained sum over $S_{1,2}$ which can be transformed with the aid of the Poisson sum formula to read

$$\langle S_+(R)\, S_-(0)\rangle$$

$$= \frac{\prod\limits_{r,i} \int_{-\infty}^{\infty} \mathrm{d}\varphi_i(r) \sum\limits_{m_i(r)=-\infty}^{\infty}{}' \exp\left[-\frac{1}{4\beta}(n_\mu(n\cdot\Delta)^{-1}Q + \epsilon_{\mu\nu i}\,\Delta_\nu\varphi_i)^2\right] \mathrm{e}^{2\pi i\, m_i\varphi_i}}{\prod\limits_{r,i} \int_{-\infty}^{\infty} \mathrm{d}\varphi_i(r) \sum\limits_{m_i(r)=-\infty}^{\infty} \exp\left[-\frac{1}{4\beta}\Sigma(\epsilon_{\mu\nu i}\,\Delta_\nu\,\varphi_i)^2\right] \mathrm{e}^{2\pi i \Sigma m_i\varphi_i}} \tag{13}$$

Now define $m_3 \equiv (n\cdot\Delta)^{-1}\,\Delta_i m_i$ and do to the $\varphi_i$ integrations. The result is

$$\langle S_+(R)\, S - (0)\rangle = \exp\left[-\frac{1}{4\beta}\,\Sigma\, Q(r)\, v(r-r')\, Q(r')\right]$$

$$\times \{\prod_{r,\mu} \sum_{m_\mu(r)=-\infty}^{\infty} \prod_r \delta_{\Delta_\mu m_\mu(r),0} \exp[-4\pi^2\beta\Sigma m_\mu(r)\, v(r-r') m_\mu(r')]$$

$$\times \frac{\exp[2\pi i \Sigma m_\mu(r)\, \epsilon_{\mu\nu\lambda}\, n_\nu\, v(r-r')\,\Delta_\lambda\, n\cdot\Delta^{-1}\, Q(r')]\}}{\prod\limits_{r,\mu} \sum\limits_{m_\mu(r)=-\infty}^{\infty} \prod\limits_r \delta_{\Delta_\mu m_\mu(r),0} \exp[-4\pi^2\beta\Sigma m_\mu(r)\, v(r-r')\, m_\mu(r')]}\,. \tag{14}$$

Eq. (14) gives the spin-spin correlation function in terms of the effective interaction between a pair of point changes immersed in a gas of monopole current loops. For $\beta$ large there are few loops and they are small, the large distance potential has the naked Coulomb form $1/R$, and the spin correlation function approaches a constant at large distances. The theory has long range order at low temperatures. It must be that above some critical temperature, the effective potential becomes linear, guaranteeing exponential falloff of the correlation function. We believe that the mechanism which forces the potential to become linear is the appearance for large enough temperature of monopole loops whose size grows without limit. We will show below that growth of loops causes a phase transition but we have not yet constructed an argument which shows that long loops imply a linear effective potential. Work on this problem is in progress.

It should be noted that in the more conventional description of the Heisenberg model our monopole loops would be vortex rings. It seems relatively clear that long vortex rings will destroy the long range spin-spin correlations.

There are two ways to approach the computation of the critical temperature in the three-dimensional Villain model. The first, a duality argument, is more rigorous but specific to this model. We look at the partition function (denominator of (14)). As one approaches the critical temperature from below the average loop size grows and the average loop is a very random walk. Thus we might expect that the Coulomb forces on any given link due to its neighbots (both on the same loop and on other loops) cancel. Thus we may approximate

$$v(r-r') \approx v(0)\, \delta_{rr'} \approx 0.253\, \delta_{rr'} \qquad \text{(ref. [9])}\,. \tag{15}$$

With this approximation our dual partition function has the same form as the untransformed function (denominator of (11)) with the replacement $1/4\beta \to 4(.253)\,\pi^2\beta$. We see that near the critical point the gas of tangled monopole loops is dual to the gas of tangled high temperature loops represented by eq. (11). In both cases the evaluation of the partition function is complicated but the complications are identical. Thus if there is a single finite temperature critical point it must be given approximately by

$$T_c \approx \sqrt{16\pi^2(2.53)} \approx 6.32\,.$$

A rigorous upper bound for the critical temperature of the 3-dimensional periodic gaussian model has been derived by Myerson [10]. It is

$$T_c \leqslant 6.2\,.$$

The corresponding bound for the full 2-component Heisenberg model is 6% above the high temperature series critical point, so our duality value is probably close but a bit high.

An alternative argument of negligible rigor but wider applicability can be given when the gas of loops is dilute. In this cases we can neglect interactions between loops because they fall off at least as fast as $1/R^3$. In fact, since the "average" loop is fairly random there is probably a substantial cancellation of multipole moments so that the interaction between distant loops falls off much faster than this. The energy of a single loop consists of a Coulomb self-energy for each point of the loop plus the mutual interactions of different points. The self-energy is proportional to the length of the loop and for long loops the mutual interaction is negligible by comparison. This is because a non-backtracking loop of $L$ steps will have a characteristic dimension $R$

$$R \sim L^x\,,$$

where $x$ is less than 0.6 (the exponent for a three-dimensional self-avoiding walk) and greater than $\frac{1}{2}$ (random walk with no restrictions). The mutual interaction for such a loop is

$$\sim R \ln R = L^x \ln L^x\,.$$

For large $L$ this is ignorable when compared to the ($\propto L$) self-energy. The number of loops of length $L$ through a given point is the number of non-backtracking random walks on the lattice which return to the origin. This number goes asymptotically like

$$f(L)\,\mu^L$$

for large $L$. Here f is a relatively slowly varying function, perhaps a power of $L$. A simple argument can be given which bounds $\mu$ between 5 and $\sqrt{3}$. Intuitively we would expect $\mu$ to be closer to the upper bound and this expectation is partially confirmed by numerical studied of self avoiding walks [11]. (There are clearly more non backtracking walks than self-avoiding walks, and the number of self avoiding walks on a lattice goes like $\mu^L$ (times a power of $L$) with $\mu$ near $q - 1$ where $q$ is the number of nearest neighbors.)

Naive balancing of the energy and entropy of a large loop now gives

$$T_\mathrm{c} \sim \frac{4\pi^2(0.253)}{\ln \mu}$$

with $\mu \leqslant 5$. Note that $\mu = 5$ gives approximate consistency with our duality result.

## 4. Four-dimensional lattice gauge theory

After the last two sections the analysis of the 4-dimensional gauge theory is rather trivial. We start from the 4-dimensional analog of eqs. (1) and (2), but the solution of the constraint equation is

$$l_{\mu\nu} = n_\mu(n\cdot\Delta)^{-1}J_\nu - n_\nu(n\cdot\Delta)^{-1}J_\mu + \epsilon_{\mu\nu\lambda\kappa}\,\Delta_\lambda l_\kappa\,, \tag{16}$$

which has a gauge freedom like eq. (12). The argument now follows that of the previous section and results in

$$\mathrm{e}^{W(C)} = \exp\left[-\frac{1}{2\beta}\Sigma\, J_\mu(r)\, v(r-r')\, J_\mu(r')\right]$$

$$\times\{\prod_{r,\mu}\sum_{m_\mu(r)}\prod_r \delta_{\Delta_\mu m_\mu(r),0}\exp[-\pi^2\beta\, m_\mu(r)\, v(r-r')\, m_\mu(r')]$$

$$\times\frac{\exp[2\pi i\Sigma m_\mu(r)\,\epsilon_{\mu\nu\lambda\kappa}\, n_\nu\, v(r-r')\,\Delta_\lambda(n\cdot\Delta)^{-1}J_\kappa(r)]\}}{(\text{same thing with } J=0)}. \tag{17}$$

This describes an electric current loop interacting with a gas of monopole current loops. This is a two-phase system and the critical temperature $T_\mathrm{c}$ which separates the phases will be estimated below using the methods applied in the previous section to the rotor model. Roughly speaking, the phase transition is of the following character. For $T < T_\mathrm{c}$ there will be few monopole loops and they will be small in spatial extent. Increasing $T$ increases the density and sizes of the loops. At $T_c$ the theory's vacuum becomes a gas of monopoles and anti-monopoles without strong correlations. Hence, the monopole number density $\nabla \cdot \boldsymbol{B}$ in the vacuum is indefinite. Then the electric field in the vacuum can be definite and presumably vanishes. If a charged impurity (quark) is placed in this vacuum its attendent electric field can spread from its source only by excluding monopole anti-monopole pairs from the space it occupies. So, if a quark is placed at position $\boldsymbol{R}$ and an anti-quark at $-\boldsymbol{R}$ the electric flux which must connect them will seek a spatial configuration to minimize the energy. It is reasonable then to expect a flux tube of length $2R$ and finite transverse extent to develop and produce a linearly rising $q\bar{q}$ potential. Roughly speaking, this phenomenon is the Meissner effect with an electric-magnetic intercharge. It has been discussed previously by Mandelstam and 't Hooft [12], although now we are in a position to quantify it. Additional work in this direction is in progress.

To estimate the critical temperature for this transition we can use the dilute gas argument of sect. 3

$$T_c = \frac{\pi^2 v(0)}{\ln \mu}, \tag{18}$$

where $\mu$ ($\approx 7$) is defined as in sect. 3, but for 4-dimensional non-back-tracking walks. $v(0)$, the value of the 4-dimensional lattice Coulomb potential at the origin has not (to our knowledge) been calculated exactly. Using the results of ref. [13] we estimate it to be about 0.38. Unfortunately we have no independent calculation of the critical temperature of the 4-dimensional lattice gauge theory to compare with (18).

We believe that the qualitative results of this section are of great importance. They indicate that the 't Hooft-Mandelstam [12] "dual Meissner effect" and the confinement mechanism of strongly coupled lattice gauge theories are one and the same phenomenon. We hope that these new insights will lead us to better understanding of non-Abelian gauge theories. In the next section we will attempt to outline some of the important features of non-Abelian gauge theories in the light of our analysis.

## 5. Non-Abelian theories

The techniques we have used in this paper do not generalize simply to non-Abelian theories. However, we believe that the intuition we have gained is useful in understanding what goes on in these theories also.

The first thing we should expect is that classical solutions with singular sources should be important in non-Abelian theories. However, in non-Abelian theories we may expect the Coulomb self energy of the monopoles to be regularized so that they survive in the continuum limit. A good example of this is the 't Hooft-Polyakov [14] monopole in the 3-dimensional Glashow model. The way in which the charged vector fields in this model regularize the monopole self-energy has been most explicitly described by Vinciarelli and Troost [15].

One can also find regularized vortices in the 2-dimensional O(3) nonlinear $\sigma$ model. These were first discovered by Susskind [16], and have the form

$$\varphi(r,\theta) = \theta(R-r)\begin{pmatrix} \dfrac{2rR}{r^2+R^2}\cos\theta \\ \dfrac{2rR}{r^2+R^2}\sin\theta \\ \dfrac{r^2-R^2}{r^2+R^2} \end{pmatrix} + \theta(r-R)\begin{pmatrix}\cos\theta \\ \sin\theta \\ 0\end{pmatrix}. \tag{19}$$

They are solutions of the equations of motion everywhere except on the circle $r = R$, and have logarithmically divergent action. A finite action vortex anti-vortex pair can be constructed.

Finally we consider the 4-dimensional non-Abelian gauge theory. As we have seen, the corresponding Abelian lattice theory contains monopole current loops. We believe that an analogous non-Abelian object is the meron of refs. [17,19]. The Minkowski-space interpretation of the monopole loop is a tunneling process in which the bare vacuum mixes with a state of a monopole anti-monopole pair. Mandelstam [18] has pointed out that the meron can be viewed as a "smoothed out" version of this process. To see this consider the singular solution of DeAlfaro et al. [19] (We use the notation of Jackiw and Rebbi [20].)

$$A_\mu = -\frac{\sigma_{\mu\alpha}\, x_\alpha}{X^2}. \tag{20}$$

Recall that this potential generates half a unit of topological charge,

$$D(x) = \frac{1}{32\pi^2}\,\mathrm{tr}\,\widetilde{F}_{\mu\nu}\,F^{\mu\nu},$$

at the origin and another half unit at infinity. By suitable conformal and gauge transformations these regions of non-vanishing $D(x)$ can be moved to arbitrary points $x_1$ and $x_2$. Eq. (20) is convenient for the discussion here since it allows us to concentrate just on one member of the monopole pair. This vector potential gives rise to field strengths

$$B_i^a = \frac{1}{x_0^2+x^2}\,(x_0^2\,\delta_i^a + x_0\,\epsilon_{ija}\,x_j + x_i x_a), \tag{21a}$$

$$E_i^a = \frac{1}{x_0^2+x^2}\,(x_0\epsilon_{ija}x_j + \delta_{ia}\,x^2 - x_i x_a). \tag{21b}$$

We perform a local gauge transformation which rotates $x_a$ into $\delta_{a3}\,|x|$ and look at time $x_0 = 0$. Then

$$E_i^3 = B_i^{1,2} = 0\ , \tag{22}$$

$$B_i^3 = \frac{x_i}{|\boldsymbol{x}|^3}\ , \tag{23}$$

$$E^{1,2} \cdot B^3 = 0\ . \tag{24}$$

At this time the configuration looks like a magnetic monopole at the origin. To see how it evolves we study the "abelian" magnetic field $B^3$ for arbitrary times

$$B^3 = \frac{\hat{x}}{x_0^2 + \boldsymbol{x}^2}\ . \tag{25}$$

The magnetic charge density is

$$\rho_m = \frac{1}{4\pi} \nabla \cdot B^3 = \frac{2x_0^2}{|\boldsymbol{x}|\,(x_0^2 + \boldsymbol{x}^2)}\ , \tag{26}$$

giving rise to a conserved magnetic charge

$$Q_\mathrm{m}(x_0) = \int \mathrm{d}^3x\, \rho_m = 1\ . \tag{27}$$

Thus the meron corresponds to a process where a diffuse magnetic charge density is created, coalesces into a monopole-antimonopole pair (above we have considered only half of the pair) and rediffuses into the vacuum. This method of creating monopoles is energy efficient, costing an action proportional to the logarithm of the distance between the pair. Of course the configuration is ultraviolet singular and must be regularized as in ref. [17].

We believe that there may be another configuration relevant to the physics of non-Abelian gauge fields. This would be regularized version of a closed string of Wu-Yang [21] monopoles.

For a long straight (time-like) piece of the string such a configuration would be time independent and given by the Wu-Yang Formula

$$A_i^a(\boldsymbol{x}, x_0) = \epsilon_{ija} \frac{x_j}{r^2}\ . \tag{28}$$

Of course to obtain a finite continuum action this configuration must be regularized, and the string must be closed. A possible regularization of the Wu-Yang solution is given by

$$A_i^a(\boldsymbol{x}) = \epsilon_{ija}\, x_j \left\{ \theta(\lambda - r) \frac{g(r)}{r^2} + \theta(r - \lambda) \frac{1}{r^2} \right\}\ . \tag{29}$$

In the appendix we demonstrate that there exists a function $g$ such that

(a) the energy of the above configuration is finite;
(b) $\epsilon_{ija} x_j \, g(r)/r^2$ satisfies the static Yang-Mills equations for $r < \lambda$;
(c) $g(\lambda) = 1$ so that $A_i^a$ is continuous and the field strengths finite.

This configuration shares with the meron and the vortex (19) the property that it satisfies the classical field equations everywhere except on a lower dimensional manifold (in this case the surface of a sphere in 3 dimensions), but its construction is a bit too *ad hoc*. One can also invent *ad hoc* descriptions of the configuration corresponding to a closed string of monopoles. Hopefully an extension of our methods to non-Abelian theories will lead to a *natural* definition of such regularized monopole loops.

The monopole string configuration is not as efficient as the meron in creating monopoles (if costs an action linear in the separation of the monopole-antimonopole pair) but it also has an entropy (corresponding to the number of configurations of the string) which grows linearly with the length of the string. It would seem that only a detailed calculation can decide which of these configurations is more important.

Non-Abelian theories have two other properties not shared by their Abelian counterparts. The first of these is perturbative asymptotic freedom whose significance can be seen by the following argument. The 2-dimensional O(2) Heisenberg model and the 4-dimensional Abelian lattice gauge theory have topological excitations which act to destroy long range correlations. However, even on the lattice there is a finite range of temperatures for which the excitations are bound together and do not effect the long range correlation. In non-Abelian theories however, perturbative asymptotic freedom implies that the effective temperature rises with the length scale and the correlation destroying excitations are always unbound.

The second new feature of (some) non-Abelian theories is the existence of instantons. There is no analog of these objects in the Abelian theories that we have studied and we have nothing to say about their role in confinement or the destruction of correlations. They do appear to provide the most natural mechanism for regularizing the vortex or monopole configurations that we have discussed.

## 6. Conclusion

We have seen that the phase transitions in Abelian lattice gauge theories may be understood in terms of monopole or vortex configurations. These arise because the variables on the lattice are angles. Of course, for an Abelian gauge theory the angle variable formalism is a matter of choice – the continuum theory does not tell us whether the gauge group is compact. For a non-Abelian theory the compactness of the group is manifest even in the infinitesimal algebra, and the angle formalism is a necessity. Thus configurations analogous to those we have discussed should be important for non-Abelian theories also.

We view the extension of the JKKN methods to non-Abelian theories to be the primary challenge posed by our work. Vortex and monopole-like configurations have already been found in non-Abelian theories by semi-classical methods. However, they are not usually exact stationary points of the continuum action and this leads to complications in the application of semiclassical methods. Clearly, a treatment analogous to the one presented here would clarify what we mean by "the meron contribution to the path integral" for arbitrary values of the coupling constant. It may also lead us to a deeper understanding of the role of instantons and to a method for incorporating all of these semi-classical excitations into a viable computational scheme.

Our aim in the present paper has been to present a qualitative picture of the phase transitions in Abelian gauge theories rather than to compute detailed properties of the models. If our sort of approach is ever to prove useful in hadron physics however we must learn how to use monopole excitations to do computations. As a warm up exercise, it would be interesting to try to compute the critical indices of the 3-dimensional plane rotor model by studying the gas of vortex rings. A related attempt has been made by Wiegel [22] in the context of liquid helium. Wiegel counted self-avoiding rather that non-back-tracking walks and neglected the interaction between vortex rings due to phonon exchange (what we have called long range Coulomb interaction in the text). Thus his failure to get the right critical indices may be remediable. We hope to return to this problem in the near future.

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## Appendix A

In his pioneering work on compact abelian gauge theories in three dimensions [23], Polyakov verified confinement explicitly by obtaining a linear potential between heavy quarks. In this appendix we obtain similar results and a physical picture of confinement using the dual form of the correlation function,

$$\langle \exp(ie \oint_{C} A_\mu \mathrm{d}x^\mu)\rangle = Z(J)/Z(0) \,, \tag{A.1}$$

where

$$Z(J) = \exp\left[-\frac{1}{2ga}\left(\frac{e}{g}\right)^2 \sum_{r,r'} J_\mu(r)\, v(r-r')\, J_\mu(r')\right]$$

$$\times \sum_{m(r)=-\infty}^{\infty} \exp\left[-\frac{2\pi}{g^2 a}\sum_{r,r'} m(r)\, v(r-r')\, m(r')\right] \exp\left[2\pi i \frac{e}{g}\sum_{r,r'} \epsilon_{\mu\nu\lambda}\, n_\mu \right.$$

$$\left. \times\, (n\cdot\Delta)^{-1}\, \Delta_\nu J_\lambda(r)\, v(r-r')\, m(r')\right], \tag{A.2}$$

where we have restored conventional field theory notation ($g$ = coupling constant with dimensions $[\mathrm{mass}]^{1/2}$, $e$ = arbitrary "charge" of external current, $x_\mu = (x, y, t)$ and repeated indices are summed.)

Consider eq. (A.2) for a simple choice of $J_\mu$. Separate a quark and an anti-quark slowly a large distance (many lattice spacings) $R$ in the $x$ direction, retain that configuration for a much longer time $T$, then bring the quarks back together as they were separated. For most of the contour (call it C),

$$J_\mu = \delta_{\mu 3}\,[\delta_{x,R/2} - \delta_{x,-R/2}]\,\delta_{y,0}\,. \tag{A.3}$$

Choosing $n_\mu$ in the $x$ direction, we compute the following quantity appearing in eq. (A.2).

$$\epsilon_{\mu\nu\lambda}\, n_\mu (n\cdot\Delta)^{-1} J_\lambda(r) = \theta_s(xt)\,\delta_{y,0}\,, \tag{A.4}$$

where $\theta_s(xt) = 1$ if $x$ and $t$ are inside C and zero otherwise. Therefore, the monopole density $\Delta_{\nu=2}\,\theta_s(xt)\,\delta_{y,0}$ generated by the external loop and coupling to lattice monopoles $m(r)$ in eq. (A.2) is a dipole sheet one lattice spacing thick of magnetic monopoles of strength $e/g$. An intuitive picture of quark confinement follows from this. Eq. (A.2) represents a dipole sheet immersed in a monopole gas. The mobile massive monopoles react to the presence of the sheet and in their most likely configuration will distribute themselves in order to screen out the magnetic field inside the dipole sheet thereby minimizing the system's energy. However, a detailed analysis [23] shows that the sheet cannot be screened completely and there is a finite energy density left behind the quarks interact through a potential which grows linearly with $R(\langle\exp(ie\oint_C A_\mu\, dx^\mu)\rangle \sim \exp(-\text{area enclosed by C}))$.

Now, following Polyakov [23], we will rewrite eq. (A.2) in the form of an effective field theory of screening. In the last term of that equation compute, near the sheet,

$$-\sum_r \Delta_y \theta_s(xt)\,\delta_{y,0}\, v(r-r') \cong \theta_s(xt)\,\theta(y)\,. \tag{A.5}$$

This is just the potential due to the dipole sheet. Now,

$$\widetilde{Z}(J) = \sum_{m(r)=-\infty} \exp\left[-\frac{2\pi}{g^2 a}\sum_{r,r'}{}' m(r)\, v(r-r')\, m(r') - \frac{2\pi}{g^2 a}\right.$$

$$\left. \times\, v_0 \sum_r m(r)^2 - 2\pi i \frac{e}{g}\sum_r \theta_s(xt)\,\theta(y)\, m(r)\right], \tag{A.6}$$

where

$$\widetilde{Z}(J) \equiv Z(J) \Big/ \exp\left[-\frac{1}{2g^2a}\left(\frac{e}{g}\right)^2 \sum_{r,r'} J_\mu(r)\, v(r-r')\, J_\mu(r')\right]$$

and $v_0 = 0.253$ determines the mass of each monopole and $\Sigma'_{r,r'}$ indicates a sum over $r$ and $r'$ omitting $r = r'$. To expedite the sum over $m(r)$ write,

$$\widetilde{Z}(J) = \prod_r \int \mathrm{d}\phi(r) \exp[-\tfrac{1}{8}\pi g^2 a \sum_r (\Delta_\mu \phi)^2] \sum_{m(r)=-\infty}^{\infty} \exp\left[-\frac{2\pi}{g^2 a} v_0 \sum_r m(r)^2 \right.$$

$$\left. - 2\pi i \frac{e}{g} \sum_r \theta_s(xt)\, \theta(y)\, m(r) + i \sum_r \phi(r)\, m(r)\right]. \tag{A.7}$$

Now choose a small lattice spacing, i.e. $(g^2a)^{-1}$ large, so that the mass of each monopole is large. Then the gas is dilute and only $m(r) = -1, 0, +1$ need be kept in the sum. Then,

$$\sum_{m(r)=-\infty}^{\infty} \exp\left[-\frac{2\pi}{g^2a} v_0 m(r)^2 - 2\pi i \frac{e}{g} \theta_s(xt)\, \theta(y)\, m(r) + i\phi(r)\, m(r)\right]$$

$$= 1 + 2\exp\left[-\frac{2\pi}{g^2a} v_0\right] \cos\left(2\pi \frac{e}{g} \theta_s(xt)\, \theta(y) + \phi(r)\right) + \begin{pmatrix}\text{terms with} \\ |m(r)| > 1\end{pmatrix}$$

$$\approx \exp\left(2\exp\left[-\frac{2\pi}{g^2a} v_0\right] \cos\left(2\pi \frac{e}{g} \theta_s(xt)\, \theta(y) + \phi(r)\right)\right). \tag{A.8}$$

So,

$$\widetilde{Z}(J) \equiv \prod_r \int \mathrm{d}\phi(r) \exp\left[-\tfrac{1}{8}\pi g^2 a \sum_r (\Delta_\mu\phi)^2 + 2\exp\left\{-\frac{2\pi}{g^2a} v_0 \sum_r \right.\right.$$

$$\left.\left. \times \cos\left(2\pi \frac{e}{g} \theta_s(xt)\, \theta(y) + \phi(r)\right)\right\}\right], \tag{A.9}$$

which is the discrete form of Polyakov's eq. (5.20) [23]. At this point our analysis parallels his, so the reader should consult the original work. From eq. (A.9) note that the coefficient of the linear potential will be proportional to $\exp(-2\pi/g^2a\, v_0)$ and vanishes non-analytically in the continuum ($a \to 0$) limit.

**Appendix B**

We will consider vector potentials of the form

$$A_i^a(x) = \epsilon_{ija}\, x_j \frac{g(r)}{r^2}\,, \tag{B.1}$$

which is the most general form invariant under combined rotations and SU(2) gauge transformations and antisymmetric under interchange of group and spatial indices. We require

(i) $A^a$ satisfies the static Yang-Mills equations (gauge group SU(2), space dimension 3) for $r < \lambda$.

(ii) $\int d^3r\, \theta(\lambda - |r|)\, F_{ij}^{a\,2} < \infty$ where $F_{ij}^a$ is the field strength tensor formed from $A$.

(iii) $g(\lambda) = 1$, so that $A$ connects continuously to the Wu-Yang solution at $r = \lambda$.

We construct the action in terms of $g$ and the variable $t \equiv \ln \lambda/r$

$$S = \frac{4\pi}{\lambda} \int_0^\infty dt\, e^t(\dot{g}^2 + 2g^3 + \tfrac{1}{2}g^4 + 2g^2)\,. \tag{B.2}$$

Symmetry considerations assure us that a stationary point of (B.2) gives rise to a solution of the Yang-Mills equations when inserted into (B.1).

The Euler-Lagrange equation for $g$ is

$$\ddot{g} + \dot{g} = 3g^2 + g^3 + 2g\,. \tag{B.3}$$

Its order can be reduced by defining

$$\dot{g}(t) = -v(g(t))\,, \tag{B.4}$$

$$t = -\int_{g_0}^{g(t)} \frac{dx}{v(x)}\,. \tag{B.5}$$

Then

$$v\frac{dv}{dg} - v = 3g^2 + g^3 + 2g\,. \tag{B.6}$$

Our boundary condition $g(r = \lambda) = 1$ reduces to

$$g_0 = 1 \tag{B.7}$$

and the requirement of finite action will be satisfied if $g$ and $\dot{g} \to 0$ sufficiently fast as $t \to \infty$, and have no singularities for finite $t$. This implies that

$$v(0) = 0\,, \tag{B.8}$$

$$\infty > v(g) > 0 \qquad \text{for } 0 < g \leqslant 1\,. \tag{B.9}$$

With these conditions $g(r)$ will vary smoothly from 0 to 1 as $r$ goes from 0 to $\lambda$. For small $g$, an approximate solution to (B.6) satisfying (B.8) is

$$v(g) \approx 2g\,, \qquad g \approx 0\,. \tag{B.10}$$

Plugging this into (B.5) we see that $g \underset{t\to\infty}{\to} e^{-2t}$ and the action is finite. For a solution obeying (B.10), $v$ and $(\mathrm{d}v/\mathrm{d}g - 1)$ are both positive for small $g$. But the right-hand side of (B.6) is positive for positive $g$ so if $v$ and $v'$ are continuous, a solution of (B.6) that goes like $2g$ for small $g$ will be positive for all $g$ in $(0, 1]$, thus satisfying (B.9). The only possible problem is a singularity in $v$ or $\mathrm{d}v/\mathrm{d}g$ but

$$\frac{\mathrm{d}v}{\mathrm{d}g} = 1 + \frac{3g^2 + g^3 + 2g}{v}\,, \tag{B.11}$$

so it is only singular when $v = 0$. But $v(g_0) = 0$ and $v(g) > 0$ for $0 < g < g_0$ implies that $\mathrm{d}v/\mathrm{d}g < 0$ for some $0 < \bar{g}_0 < g_0$ which contradicts (B.11). Thus $\mathrm{d}v/\mathrm{d}g$ is non-singular on $(0,1]$ and so is $v = \int_0^g \mathrm{d}v/\mathrm{d}g$.

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