ORIGINS OF GLOBAL ANOMALIES IN QUANTUM MECHANICS

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Simple and tractable examples of abelian and non-abelian gauge systems with global anomalies are presented in quantum mechanics. Explicit calculations are done both in the path-integral and hamiltonian formalism. Algebraic criteria are given for the existence of global gauge anomalies. These criteria are applied for every gauge group and many representations. The inconsistency of theories with a global gauge anomaly is discussed.

1. Introduction

Local anomalies are old companions in studying quantum field theories [1]. Global anomalies, on the other hand, are rather recent, appearing in gauge [2] and gravitation [3] field theories as well as in string [4] theories.

Anomalies occur when symmetries of the classical system cannot be maintained in the quantum theory. Starting with the path-integral formulation of gauge theory we first obtain an effective action by integrating out the fermionic fields using some regularization. It may then turn out that certain symmetries cannot be jointly incorporated into the regulated action. In a hamiltonian formulation this freedom in choosing a regularization scheme shows up as an ordering ambiguity, when classical variables are replaced by their quantum counterparts. This ambiguity may not be enough to allow a quantum realization of all classical symmetries. Both these ingredients appear already in quantum mechanical systems. Due to the simple
structure of these models, the mechanism leading to the anomalies can be easily identified.

Global anomalies arise when the effective action fails to be invariant under a large gauge transformation, one that is disconnected from the identity. The anomaly can take two forms. Either the gauge invariance is inherently broken, or that the gauge invariance can be restored at the expense of breaking some global classical symmetry. Quantum mechanical examples exist for both cases. Moreover, it will be shown that in the case of a dynamical gauge field, a system with a global gauge anomaly is not consistent.

We start our analysis with a few general remarks. Consider the 0-space + time euclidean action
\[ S = \int dt \left[ -\bar{\theta}_e \frac{\partial}{\partial t} \theta_e + \bar{\theta}_k \lambda^m_{kl} \theta_l A^m(t) \right], \] (1.1)
with \( \theta_k \) fermionic variables transforming under some representation of the gauge group whose generators are the anti-hermitian matrices \( \lambda^m \). When the fermions \( \theta_k \) are complex or pseudoreal, namely \( \bar{\theta} \neq \theta \), there will be no gauge anomaly. This is a consequence of the fact that one can always regularize the system by adding Pauli-Villars fields \( P_k \) in the same representation with the gauge-invariant mass term
\[ M \sum_k \bar{P}_k P_k. \] (1.2)
Although this term may induce an anomaly in some global symmetry, gauge anomalies can be avoided. An anomaly in the gauge symmetry may occur only for a real system with the action
\[ S = \frac{1}{2} \int_0^T dt \left[ -\bar{\psi}_i \frac{\partial}{\partial t} \psi_i + \psi_k \lambda^m_{kl} \psi_l A^m(t) \right], \] (1.3)
with \( \psi_i \) being real fermions transforming under the real representation with the antisymmetric generators \( \lambda^m \). As a corollary of this fact it follows that in any case the gauge anomaly could manifest itself at most by a change of sign of the generating functional when a topologically non-trivial gauge transformation is performed. Indeed, generally, the system (1.3) does not allow for a gauge-invariant regularization, and has to be regularized by a possibly gauge-non-invariant regularization scheme \( \Lambda \), giving a generating function
\[ Z^{(1)}_\Lambda = \int \prod \psi_i \exp[S_\Lambda]. \] (1.4)

Following Alvarez-Gaumé and Witten [5], one can add another set of real fermions \( \phi \) in the same representation and obtain a new system which can be
regularized in two ways. First one can use the previous non-invariant regularization \( \Lambda \) on both species of fermions \( \psi \) and \( \phi \). The generating function for the new doubled system is, then,

\[
Z_{\psi, \phi} = Z_{\Lambda}^2.
\]  

(1.5)

On the other hand, the same doubled system can be described by the complex fermionic variables

\[
\theta_k = \sqrt{\frac{1}{2}} (\psi_k + i\phi_k)
\]  

(1.6)

and may be regularized by the gauge-invariant Pauli-Villars mass term of (1.2), giving the generating function \( Z_\theta(A) \). For the doubled system we have then two partition functions \( Z_\theta \) and \( Z_{\psi, \phi} \) resulting from two different regularizations of the same system. In the limit of a very high regulator scale the two functions differ at most by an exponential of a local functional of the background field \( A \)

\[
Z_{\psi, \phi} = Z_\theta e^{F(A)}.
\]

By adding the local expression \(-\frac{1}{2}F(A)\) to the original \( \psi \) action (1.3) as a counterterm, one gets for the modified system

\[
Z_{\Lambda}^2 = Z_\theta,
\]

which is gauge invariant by construction. Since \( Z_{\Lambda}^2 \) is invariant, \( Z_\Lambda \) can change at most by a sign under a globally non-trivial gauge transformation.

For a non-trivial topology to exist we let \( t \) in (1.3) vary in the interval \([0, T]\) which is taken to be a circle \( S^1 \). Possible candidate representations of the fermions in (1.3) for producing anomalies are real representations which are simply connected and whose \( \pi_1 \) group contains elements of even order, i.e. elements which give the identity only when raised to an even power (elements of odd order cannot induce a change of sign in the generating functional). Since there is a correspondence between the elements of the \( \pi_1 \) group of a representation and the elements of the center of the original covering group we have to look for real representations of groups with even order elements in their center.

In sect. 2 the \( U(1) \) case will be studied using the path integral method. An anomaly will be found in either some discrete global symmetry or in a topologically non-trivial gauge transformation. In sect. 3 the same partition function is calculated in the hamiltonian formulation. In both sections we demonstrate that the system cannot be consistently quantized with dynamical gauge fields in the presence of a global gauge anomaly.

Sect. 4 deals with the non-abelian case using a hamiltonian approach. The \( O(N) \) case was treated by Witten [4] in the lagrangian formalism, where it was related to
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the problem of the existence of spin structures on manifolds. For groups and representations satisfying the conditions discussed above, one may find genuine $Z_2$ gauge anomalies which cannot be traded for a global symmetry, in which case the gauge field cannot be consistently quantized. For every relevant group we shall study the anomaly properties of at least one real representation for all possible structures of the $\pi_1$ group. We shall also give rules on how to calculate the anomaly of a general representation. The search for global anomalies has been turned into an algebraic problem.

2. Anomalies in an abelian system-lagrangian formulation

Consider the lagrangian

$$\mathcal{L} = \tilde{\psi}(t)[i \partial_t + A(t)] \psi(t).$$

(2.1)

Note that in $(0 + 1)$ dimensions $\gamma_0 = 1$, and thus $\tilde{\psi} = \psi^\dagger$. In addition to its gauge invariance under

$$\Psi(t) \rightarrow e^{iA(t)}\Psi(t), \quad \bar{\psi}(t) \rightarrow \bar{\psi}(t)e^{-iA(t)}, \quad A(t) \rightarrow A(t) + \partial_t A(t),$$

(2.2)

with $e^{iA(t)}$ a periodic function in $[0, T]$, the lagrangian is invariant under a discrete symmetry, charge conjugation, under which

$$\Psi(t) \leftrightarrow \bar{\Psi}(t), \quad A(t) \rightarrow -A(t).$$

(2.3)

Corresponding to any given background gauge field $A(t)$ there is a gauge invariant quantity

$$W = \exp\left(i \int_0^T A(t) dt\right).$$

By a small gauge transformation, i.e. one which is continuously connected to the identity, $A$ can be brought into a time independent form $\tilde{A}$

$$\tilde{A} = \frac{1}{T} \int_0^T dt A(t).$$

(2.4)

A large gauge transformation $A(t) = (2\pi n/T)t$ then shifts $\tilde{A}$ by $2\pi n/T$ leaving only the fractional part of $\tilde{A}T/2\pi$ gauge invariant. In particular, if $\tilde{A}T/2\pi$ is not an integer, the background field cannot be gauged to zero. The euclidean partition function is, in the gauge (2.4),

$$Z(\tilde{A}) = \int d\Psi(t) d\bar{\Psi}(t) \exp\left[\int dt \bar{\Psi}(t)(-\partial_t + i\tilde{A})\Psi(t)\right].$$

(2.5)
The potential $A$ is, at this stage, taken to be a classical background.

$$Z(A) = \det[-\partial_t + iA]. \tag{2.6}$$

To give a meaning to (2.6) we have to specify a regularization scheme. Let us choose, first, a gauge-invariant Pauli-Villars regularization. It turns out that to maintain $Z(A)$ finite, several Pauli-Villars ghosts are needed. Specifically, replace (2.6) by

$$Z_{PV}(A) = \lim_{M_1, M_2 \to \infty} \left[ \det(-\partial_t + iA) \right] \left[ \det(-\partial_t + iA + M_1) \right]^{\alpha_1} \times \left[ \det(-\partial_t + iA + M_2) \right]^{\alpha_2}. \tag{2.7}$$

$Z_{PV}$ is finite if

$$1 + \alpha_1 + \alpha_2 = 0,$$

$$\alpha_1 M_1 + \alpha_2 M_2 = 0. \tag{2.8}$$

Explicitly,

$$Z_{PV}(A) = \lim_{M_1, M_2 \to \infty} \lim_{N_1, N_2 \to \infty} \prod_{n=-N_1}^{N_2} \left[ \frac{2\pi}{T} \left( n + \frac{1}{2} + \tilde{A} \right) \right]^{\alpha_1} \times \left[ \frac{2\pi}{T} \left( n + \frac{1}{2} + \tilde{A} + iM_2 \right) \right]^{\alpha_2}. \tag{2.9}$$

Antiperiodic boundary conditions have been chosen in (2.9) such that $Z(A)$ represents $\text{Tr} e^{-TH}$. The resultant anomaly is, however, insensitive to the choice of boundary conditions.

We have

$$Z_{PV}(A) = \lim_{m_i \to \infty} \lim_{N_1 \to \infty} \left[ \frac{\Gamma(N_2 + \frac{1}{2} + a)}{\Gamma(-N_1 + \frac{1}{2} + a)} \right]^{\alpha_1} \times \left[ \frac{\Gamma(N_2 + \frac{1}{2} + a + im_2)}{\Gamma(-N_1 + \frac{1}{2} + a + im_2)} \right]^{\alpha_2},$$

where $a = \tilde{A}T/2\pi$, $m_i = M_i T/2\pi$. When the conditions (2.8) are satisfied, the limits $N_{1,2} \to \infty$ exist and are independent of the relative rate of the approach of $N_1$ and $N_2$ to infinity.

Using standard identities one obtains

$$Z_{PV} = \lim_{m_i \to \infty} (\cos \pi a) [\cos \pi (a + im_1)]^{\alpha_1} [\cos \pi (a + im_2)]^{\alpha_2}$$

$$= 1 + e^{2\pi i a} = 1 + e^{i\tilde{A}T}.$$ \tag{2.10}
By construction, $Z_{PV}$ is indeed invariant under the large gauge transformation $\tilde{A} \rightarrow \tilde{A} + 2\pi k/T$, $k$ integer. It is also invariant by construction under local gauge transformations as $\tilde{A}$ is locally gauge invariant. On the other hand, charge conjugation symmetry (2.3), which would imply

$$Z(\tilde{A}) = Z(-\tilde{A})$$

(2.11)

is lost despite its presence in the classical unregulated lagrangian. Indeed, the system has a charge conjugation anomaly. One can attempt to restore the symmetry by adding an $A$-dependent counterterm to the lagrangian of the form

$$L_{\text{ct}}^\text{euclid} = -\frac{1}{2} \int_0^T iA \, dt.$$  

(2.12)

The restoration of charge conjugation is at the expense of invariance under the large gauge transformations, since (2.12) is not invariant under the global gauge transform

$$A \rightarrow A + \partial_t \Lambda(t),$$

(2.13)

where

$$\Lambda(t) = \frac{2\pi kt}{T}$$

for $k$ an odd integer. Note that for an even value of $k$ gauge invariance is left intact. Thus for two fermionic flavors the factor $\frac{1}{2}$ in eq. (2.12) is replaced by 1 and no damage is involved in restoring charge conjugation. The lagrangian with the restoring term is not classically charge conjugation invariant. However, it becomes so after taking into account the anomalous influence of quantum fluctuations.

Instead of the gauge invariance preserving Pauli-Villars regularization one could regularize the system in a charge conjugation invariant way. Fourier transforming the lagrangian

$$\int L_E \, dt = \sum_n \overline{\Psi}_{-n - 1/2} \left[ \frac{2\pi i}{T} \left( n + \frac{1}{2} \right) + i\tilde{A} \right] \Psi_{n + 1/2},$$

(2.14)

where

$$\Psi(t) = \sum_n \Psi_{n + 1/2} \exp\left( -\frac{2\pi}{T} i(n + \frac{1}{2})t \right), \quad \Psi(t) = \sum_n \overline{\Psi}_{n + 1/2} \exp\left( -\frac{2\pi}{T} i(n + \frac{1}{2})t \right).$$

(2.15)

Under charge conjugation (2.3)

$$\begin{cases}
\tilde{A} \rightarrow -\tilde{A} \\
\Psi_{n + 1/2} \leftrightarrow \overline{\Psi}_{n + 1/2}
\end{cases}.$$
Eq. (2.14) will be charge conjugation invariant provided that the change of variables 
\((n + \frac{1}{2}) \leftrightarrow -(n + \frac{1}{2})\) is possible. A charge conjugation regularization is therefore one 
in which \(N_1\) and \(N_2\), the cutoffs of the product over eigenmodes, are related by

\[ N_2 + \frac{1}{2} = N_1 - \frac{1}{2}, \]

\[
\frac{Z_c(\tilde{A})}{Z_c(0)} = \lim_{N \to \infty} \prod_{n = -N}^{N - 1} \frac{(n + \frac{1}{2}) + a}{n + \frac{1}{2}} = \lim_{N \to \infty} \prod_{n = 0}^{N - 1} \left( 1 - \frac{a^2}{(n + \frac{1}{2})^2} \right)
\]

\[= \cos(\pi a) = \cos \frac{1}{2} \tilde{A} T. \tag{2.16} \]

This is indeed invariant under \(\tilde{A} \to -\tilde{A}\). However, under the large gauge transformation \(\tilde{A} \to \tilde{A} + 2\pi m/T\), (2.16) changes the sign for odd \(m\).

We see that there is indeed a clash between charge conjugation symmetry and gauge invariance. One cannot maintain both but one has to choose between them. In higher dimensions, in order to have a consistent theory of dynamical gauge fields, one has to sacrifice every global symmetry which clashes with gauge invariance. Note that the ratio of the two differently regularized partition functions appearing in eqs. (2.10), (2.16) is an exponential of a local, though not gauge-invariant, functional of the field \(A\), that is the term in eq. (2.12).

Even in our zero-dimensional case one can imagine path integrating over the gauge field \(A\). Had we chosen the charge conjugation (CC)-invariant regularization, the dependence of the effective action on \(A\) would be given by (2.16) and the resulting partition function would be zero. The theory is inconsistent. In a gauge-invariant regularization, however, the partition function is necessarily a periodic function of \(\tilde{A}\) with a period \(2\pi/T\). In general it is a sum of harmonics. In the absence of a zero harmonic the path integral will again vanish. By adding to the action a Chern-Simons term of the form

\[ L_{CS} = \int_0^T A \, dt, \]

one picks up a contribution to the partition function from the \(nth\) harmonic. Note that the coefficient \(n\) in front of \(L_{CS}\) is quantized because of gauge invariance. One cannot really fix in a similar way the CC-invariant partition function since a Chern-Simons term of the form \(\frac{1}{2} \int_0^T A \, dt\) will restore the gauge invariance of (2.16) at the expense of breaking CC.

3. Anomalies – a hamiltonian formulation

In the path integral formulation the anomaly resulted from the necessity to regularize the fermionic determinant. In the hamiltonian formulation the origin of
the anomaly is traced to the ambiguities inherent in the ordering of operators while passing from the classical to the quantum formulation.

The most general euclidean hamiltonian corresponding classically the lagrangian (2.1) is

\[ H = i\tilde{A}(\alpha\psi^\dagger\psi + \beta\psi\psi^\dagger), \quad \alpha = 1 + \beta. \]  

In the quantum theory \( \psi, \psi^\dagger \) satisfy the anti-commutation relations:

\[ \{\psi^\dagger, \psi\} = 1, \quad \{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0. \]

Thus the hamiltonian can be written as:

\[ H = i\tilde{A}(\psi^\dagger\psi + \beta), \]  

where \( \beta \) parametrizes the ordering ambiguity.

Under charge conjugation \( \psi \leftrightarrow \psi^\dagger, A \rightarrow -A \); thus \( H \) will be invariant provided we choose \( \beta = -\frac{1}{2} \). \( H \) being the hamiltonian of a single fermionic oscillator has two energy levels. The partition function is thus:

\[ Z(\tilde{A}) = \text{Tr} e^{-TH} = e^{-i\tilde{A}\beta}\left[1 + e^{-i\tilde{T}}\right]. \]

For a general \( \beta \), the partition function (being a function of \( \tilde{A} \)) is always locally gauge invariant, however it is neither globally gauge invariant nor charge conjugation invariant. Only for \( \beta = -\frac{1}{2} \) is \( Z(\tilde{A}) \) charge conjugation invariant, in which case

\[ Z_c(\tilde{A}) = 2\cos\frac{1}{2}\tilde{A}T. \]

\( Z_c \) is an even function of \( \tilde{A} \), but is not invariant under the global gauge transformations \( \tilde{A} \rightarrow \tilde{A} + 2\pi k/T \) for odd \( k \).

This partition function is identical to that obtained from the path integral formulation with a charge conjugation invariant regularization (eq. (2.16)). A globally gauge-invariant partition function is obtained for any integer \( \beta \), where the partition function for \( \beta = 0 \) coincides with the one obtained by the Pauli-Vilars method (eq. (2.10)). Its imaginary part is not charge conjugation invariant.

Up to this stage we have maintained an even-handed policy towards global gauge and charge conjugation anomalies. However, it turns out, again, that a theory with a dynamical gauge field will be inconsistent if the gauge symmetry is not maintained. The dynamics of the gauge field manifests itself in one dimension through Gauss' law:

\[ \psi^\dagger\psi + \beta = 0. \]  

From the commutation relations (3.2) it follows that \( \psi^\dagger\psi \) has eigenvalues zero and
one. Therefore, Gauss' law cannot be imposed for non-integer values of $\beta$, in particular for the CC-invariant value $\beta = -\frac{1}{2}$. For two species the difficulty disappears.

For one species we have seen that charge conjugation invariance has to be abandoned in favor of gauge invariance. A general gauge invariant action may involve also a Chern-Simons term $i \int \mathcal{A} \, dt$. In the presence of such a term eq. (3.5) changes into:

$$\psi^\dagger \psi + \beta = n.$$  \hfill (3.6)

For any integer $\beta$ there exists integer values of $n$ for which this equation is satisfied. Note that the gap between the one-particle and the ground state of (3.3), which is $\tilde{A}$, is not invariant under large gauge transformations. This is not a problem when $A$ is a dynamical variable, since then $H = 0$ on all states.

4. Non-abelian global anomalies – hamiltonian formalism

Consider the system (1.3) with $\Psi_i, \ i = 1 \ldots n$ being real fermionic variables transforming under the real representation $r$ of the gauge group. The gauge-invariant lagrangian is given by

$$L = \frac{1}{2} i \Psi_i \left[ \partial_t \delta_{ij} + A_{ij}(t) \right] \Psi_j,$$  \hfill (4.1)

where $A_{ij}(t) = \lambda_{ij}^r A^m(t)$ is a real antisymmetric generator of the group in the representation $r$. The time and gauge dependent euclidean hamiltonian is

$$H(t) = \frac{1}{2} A_{ij}(t) \Psi_i \Psi_j,$$  \hfill (4.2)

where the operators $\Psi_i$ fulfill the anticommutation relations

$$\{ \Psi_i, \Psi_j \} = \delta_{ij}.$$  \hfill (4.3)

The partition function is

$$Z [A(t)] = \text{Tr} \ T \exp \left( - \int_0^T H(t) \, dt \right).$$  \hfill (4.4)

Apply now a gauge transformation $\Lambda(t)$ with $\Lambda(0) = \Lambda(T)$ to the background gauge field.

$$A^\Lambda(t) = \Lambda^{-1} A \Lambda + \Lambda^{-1} \partial_t \Lambda,$$

$$Z(A^\Lambda) = \text{Tr} \ T \exp \left\{ - \frac{1}{2} \int_0^T dt \left[ \Lambda_{ij}^{-1} A_{\mu} \Lambda_{jk} + \Lambda_{ij}^{-1} \partial_t \Lambda_{jk} \right] \Psi_i \Psi_k \right\}.$$  \hfill (4.5)
The Hilbert space $S$ of states upon which the operators $\Psi_i$ act also carries a generally reducible representation, $R$, of the gauge group. Namely

$$\Lambda_{ij} \Psi_j = U(\Lambda) \Psi_i U^\dagger(\Lambda),$$

(4.6)

$U(\Lambda)$ being unitary operators in the Hilbert space $S$ representing the gauge group. The representation $R$ is generated by the operators $\Lambda^m = \frac{1}{2} \Lambda_{ij} \Psi_i \Psi_j$, i.e. if $\Lambda = e^{a^m \Lambda^m}$ then $U(\Lambda) = e^{-a^m \Lambda^m}$. The $t$ derivative of (4.6) gives

$$[U \partial_t U^\dagger, \Psi_j] = \left[ (\partial_t \Lambda^{-1}) \Lambda \right]_{jk} \Psi_k,$$

which is solved by

$$U \partial_t U^\dagger = -\frac{1}{2} \left[ (\partial_t \Lambda^{-1}) \Lambda \right]_{jk} \Psi_j \Psi_k = \frac{1}{2} \left[ \Lambda^{-1} \partial_t \Lambda \right]_{jk} \Psi_j \Psi_k \quad (4.7)$$

due to the anticommutation relations (4.3). Eq. (4.5) now becomes

$$Z(\Lambda^A) = \text{Tr} \, T \exp \left( \int_0^T dt \left[ -U(t) H U^\dagger(t) - U(t) \partial_t U^\dagger(t) \right] \right).$$

The operator $O(S) = T \exp(\int_0^S dt \left[ -U(S) H U^\dagger(S) - U(S) \partial_s U^\dagger(S) \right])$ solves the equation

$$\frac{\partial}{\partial S} O(S) = \left[ -U(S) H U^\dagger(S) - U(S) \partial_s U(S) \right] O(S),$$

with the initial condition $O(0) = 1$. The operator $U(S) T \exp(-\int_0^S H dt) U^\dagger(0)$ solves the same equation with the same condition. Therefore

$$O(S) = U(S) \left[ T \exp \left( \int_0^S H dt \right) \right] U^\dagger(0),$$

$$Z(\Lambda^A) = \text{Tr} \, \left[ U(T) \left[ T \exp \left( \int_0^T H(t) dt \right) \right] U^\dagger(0) \right]. \quad (4.8)$$

Now, if the closed curve $\Lambda(t)$ with $\Lambda(0) = \Lambda(T)$ is contractible in the representation $R$ of the group, then, since locally the representations $r$ and $R$ are isomorphic, $\Lambda(0) = \Lambda(T)$ implies $U(0) = U(T)$. The transformation (4.8) induced by the gauge change $\Lambda(t)$ is a similarity transformation on the evolution operator $T \exp(-\int H dt)$ which does not affect its trace. Hence $Z(\Lambda^A) = Z(\Lambda)$. If, however, the curve $\Lambda(t)$ is not contractible, it may happen that while $\Lambda(T) = \Lambda(0)$, $U(T)$ differs from $U(0)$ by an element of the center of the covering group, which, in the representation $r$ is represented by the identity. The criterion for the occurrence of an anomaly is, therefore, the existence of a center element which is non trivial in $R$ but is
represented trivially by the identity in \( r \). This question can be expressed in terms of the weights of the two representations [6]. Namely, if in the weight lattice of the group, the weights of the representation \( R \) are in the same equivalence class as those of the representation \( r \) modulo the root lattice then they have the same center. Our task in studying the anomaly for \( \psi_i \) in (4.1) transforming under \( r \), is to find the weights of the corresponding representation \( R \) and check their relation to those of \( r \) modulo the root lattice.

Let \( \lambda^m, 1 \leq m \leq \text{rank of the gauge group}, \) span the Cartan subalgebra for the Lie-algebra of the group. As in (4.1) \( \lambda^m \) are matrices in the real representation \( r \). They can be brought into the form

\[
\lambda^m = \begin{pmatrix}
0 & a_1^m & 0 \\
-a_1^m & 0 & a_2^m \\
0 & -a_2^m & 0 \\
& & & \ddots
\end{pmatrix},
\tag{4.9}
\]

\( a_k \) being \( r \)-weights. Define the creation and annihilation operation

\[
a_k = \sqrt{\frac{1}{2}} \left( \psi_{2k-1} + i\psi_{2k} \right), \quad a_k^* = \sqrt{\frac{1}{2}} \left( \psi_{2k-1} - i\psi_{2k} \right).
\tag{4.10}
\]

In the representation \( R \) the operators \( \Lambda^m \) corresponding to \( \lambda^m \) of (4.9) are

\[
\Lambda^m = \sum_{k=1}^{[d_r/2]} a_k^m \left( a_k a_k^* - \frac{1}{2} \right),
\tag{4.11}
\]

where \( d_r \) is the dimension of the representation \( r \). The weights of \( R \) have the form

\[
\delta = \frac{1}{2} \sum \pm a_k,
\tag{4.12}
\]

corresponding to all possible occupation configuration of the \([d_r/2]\) fermionic levels denoted by \( k \). The highest weight is

\[
\delta = \frac{1}{2} \sum a_k,
\tag{4.13}
\]

with all the \( a_k \) in (4.9) taken to be positive weights. There will be an anomaly, if \( \delta \) is not a combination with integer coefficients of the weights of \( r \) and the root lattice. There is no need to check the other weights of \( R \) since by (4.12) they all differ from \( \delta \) by an \( r \) weight. Note that \( 2\delta \) is always in the weight lattice of \( r \), which confirms the previous observation that only a \( Z_2 \) anomaly is possible.

If a weight \( \omega \) is a combination of \( r \) weights with integer coefficients whose sum is an even number then it can be expressed as the sum of two weights \( \omega_1 \) and \( \omega_2 \) which are combinations of \( r \) weights with an odd sum of coefficients: \( \omega = \omega_1 + \omega_2 \).
The weight $\omega$ will appear in the product of two irreducible real representations, one which contains $\omega$, and the other containing $\omega_2$. As real representations their center element is represented by $-1$. Their product will have no center. The weight $\omega$ is therefore in the root lattice and the corresponding representation is contained in the product of some finite number of copies of the adjoint representation. If there is no anomaly, $\delta$ is a combination of $r$ weights and roots. By (4.12) $\delta - \alpha_k$ is also a weight in $R$. Either $\delta$ or $\delta - \alpha_k$ have an even sum of coefficients of $r$ weights. Therefore the criterion for the absence of anomaly can also be stated as the demand that the representation $R$ will contain irreducible representations which can be built from the adjoint representation alone.

We will now check the anomaly for various representations of all Lie groups. Consider first the adjoint representations of SU($2n$), SO($n$), Sp($n$) and E(7). The rest of Lie groups do not possess a center with even order elements. For the adjoint representation $\delta$ is given by a general formula [6] as the sum of fundamental dominant weights. A general formula does not seem to exist, though, for an arbitrary representation*. We will calculate $\delta$ therefore in most cases explicitly.

**SU($2n$) adjoint representation.** The positive roots are $e_i - e_j$, $1 < i < j < 2n$ where $e_i$ are orthonormal vectors in $2n$-dimensional euclidean space. By (4.13)

$$\delta = \frac{1}{2} [(2n-1)e_1 + (2n-3)e_2 + \cdots - (2n-1)e_{2n}] .$$

$\delta$ is a combination of $e_i$ with non-integer coefficients and therefore is not in the same coset as the adjoint representation. Hence there is an anomaly for every $n$.

**Sp($n$) adjoint representation.** The positive roots are $e_i \pm e_j$, $1 \leq i < j \leq n$ and $2e_i$

$$\delta = ne_1 + (n-1)e_2 + \cdots + e_n .$$

The lattice generated by the roots consists only of the points whose sum of coordinates is even. For $\delta$ this sum is $\frac{1}{2}n(n+1)$. There will be an anomaly when this number is odd. This occurs for $n = 4k + 1$ or $4k + 2$. Otherwise, when the sum of coefficients is even, it is in the root lattice. Although the non-trivial topological structure does allow for an anomaly, this option is left unused here. This occurs also in higher dimensions [2].

**SO($2n$) adjoint representation.** The positive roots are $e_i \pm e_j$, $1 \leq i < j \leq n$

$$\delta = (n-1)e_1 \pm \cdots \pm e_{n-1} .$$

By the same argument as above there is an anomaly when the sum of coordinates of the vector $\delta$ is odd. $\frac{1}{2}n(n+1)$ odd means $n = 4k + 2$ or $4k + 3$.

* This generalisation of $\delta$ was used by the authors in ref. [7] for proof of the Freudenthal-de Vries formula.
SO$(2n + 1)$ adjoint representation. The positive roots are $e_i \pm e_j, \ 1 \leq i < j \leq n$ and $e_i$:

$$\delta = \frac{1}{2} \left[ (2n - 1) e_1 + (2n - 3) e_2 + \cdots + e_n \right].$$

Non-integer coefficients appear in $\delta$, therefore it is not in the root lattice. There is an anomaly for any $n$.

$E(7)$ adjoint representation. Using the general formula mentioned above [6] we obtain

$$\delta = 17 e_1 + \frac{49}{2} e_2 + 33 e_3 + 48 e_4 + \frac{23}{2} e_5 + 26 e_6 + \frac{22}{2} e_7,$$

which does not belong to the root lattice. Hence the system is anomalous.

In addition to the adjoint representation there are other simple representations which are real and have a non-trivial $\pi_1$ which may be different from the $\pi_1$ of the adjoint representation. We shall study the vector representation of $O(n)$ and the $2n$ box antisymmetric representation of $SU(4n)$, and the chiral spinor representation of $SO(8n)$.

$SO(n)$ vector representation. The positive weights are $e_i, \ i = 1 \ldots n$

$$\delta = \frac{1}{2} \sum e_i,$$

which, due to the appearance of non-integral coefficients, is not an integral combination of $e_i$ and $e_i \pm e_j$. There is an anomaly for any $n$.

$SU(4n), 2n$ box antisymmetric representation. The positive roots are $\frac{1}{2}[e_1 \pm e_2 \pm \cdots \pm e_{2n}]$ with $(2n) -$ signs and $(2n - 1) +$ signs. These we call form (4.14).

$$\delta = \frac{1}{2} \left[ \left( \frac{4n - 1}{2n} \right) e_1 - \frac{1}{4n - 1} \left( \frac{4n - 1}{2n} \right) (e_2 + e_3 + \cdots + e_{4n}) \right].$$

If $\left( \frac{4n - 1}{2n} \right)$ is an odd number, then the coordinates of $\delta$ are non-half integers and hence $\delta$ is not an integral combination of roots and weights of the form (4.14) and there is an anomaly. This happens for $n = 2^k$. Note that $\delta = \frac{1}{2} \left( \frac{4n - 1}{2n} \right) 4n/(4n - 1)$ $[(4n - 1)/4n) e_1 - (1/4n) e_2 - \cdots -(1/4n) e_{4n}] = \left( \frac{4n - 3}{2n - 2} \right) \omega_f$ where $\omega_f$ is the highest weight of the fundamental representation. For $n \neq 2^k$, $\left( \frac{4n - 1}{2n} \right) = 2m$ is even and then the number

$$\left( \frac{4n - 3}{2n - 2} \right) = \frac{1}{4} \left( \frac{4n - 1}{2n} \right) \frac{4n}{4n - 1} = \frac{2mn}{4n - 1}$$

is divisible by $2n$ since it is an integer and $2n$ and $4n - 1$ have no common divisor. $\delta$ is therefore a multiple of $2n$ times a fundamental weight and hence belong to the same coset with respect to the center as the $2n$ box representation. Thus there is no anomaly for $n \neq 2^k$. 
Chiral spinor representation of SO(8n). The covering group of SO(k) which is simply connected is spin(k) which, for even k has the faithful representation $S_L + S_R$ where $S_{L(R)}$ is the left(right)-handed spinor representation. For k divisible by 4, the center of this group is $Z_2 \times Z_2$. The chiral spinor representation, $S_L$, as an irreducible representation, has as its center only $Z_2$. Since it does not represent the full center faithfully, its $\pi_1$ groups is $Z_2$. If further, k is divisible by 8, the representation $S_L$ is also real. Being real and non-simply connected there is a potential anomaly. The positive weights of $S_L$ for SO(8n) are $\frac{1}{2}[e_1 \pm e_2 \pm \cdots \pm e_{4n}]$ with an even number of $-$ signs. Hence

$$\delta = 2^{4n-4}e_1.$$ 

Any combination of the weights of $S_L$ and the roots $e_i \pm e_j$ has an even sum of coefficients. Therefore the case $n = 1$ for which $\delta = e_1$ is anomalous. The two chiral spinor representations of SO(8) give, then, anomalous theories. This is expected, since as proven above the vector representation gives an anomalous theory, and the vector and spinor are related by the triality property of SO(8).

The above calculations cover representations of every possible coset with respect to the center. Any other real representation of the above groups is contained in the tensor product of two smaller representations at least one of which belongs to the same coset as the adjoint representation, i.e. has no center. To deal with a general representation the following formula may be useful. Let $r_1$ and $r_2$ be two real representations and suppose that $r_1 \times r_2$ is equivalent to the adjoint representation. Then

$$\delta_{r_1 \times r_2} = \frac{1}{2} \sum_{\omega_1 + \omega_2 > 0} (\omega_1 + \omega_2)$$

$$= \frac{1}{2} \sum_{\omega_1 + \omega_2 > 0} (\omega_1 + \omega_2) + \frac{1}{2} \sum_{\omega_2 = 0} \omega_1 + \frac{1}{2} \sum_{\omega_2 > 0} \omega_2,$$

(4.15)

where $\omega_1$ and $\omega_2$ are the weights of $r_1$ and $r_2$. In the first term of the sum, take a typical term $\frac{1}{2}(\omega_1 + \omega_2) > 0$. If $\omega_1 > \omega_2$ then there is another term $\frac{1}{2}(\omega_1 - \omega_2)$ summing together to give $\omega_1$. The same is true if $\omega_1 = \omega_2$. If $\omega_2 > \omega_1$ the additional term $\frac{1}{2}(\omega_2 - \omega_1)$ adds up to $\omega_2$. The first term in (4.15) is therefore a combination of weights of the same coset as $r_2$, hence $r_1 \times r_2$, with integer coefficients. As far as the anomaly is concerned we have:

$$\delta_{r_1 \times r_2} = n_1 \delta_2 + n_2 \delta_1,$$

(4.16)

where $n_1$ and $n_2$ are the multiplicities of the zero weight in the representations $r_1$ and $r_2$, respectively. The equivalence relation in (4.16) denotes equivalence modulo the weight lattice generated by the weights of $r_1 \times r_2$ (or $r_2$).
When the gauge field is made dynamical one can recognise the trouble in an anomalous theory by trying to impose Gauss' law on the states of the system. Choose in the lagrangian (4.1) a gauge in which $A$ is $t$ independent. This is done as follows. Define

$$U(t) = T \exp \left( \int_0^t A(t') dt' \right).$$

Let the matrix $\tilde{A}$ be defined by $U(T) = e^{-\tilde{A}T}$. Then under the gauge transformation

$$A(t) \rightarrow \Lambda A \Lambda^{-1} + \Lambda \partial_t \Lambda^{-1},$$

with $\Lambda(t) = e^{-\tilde{A}t} U^\dagger(t)$ which satisfies the periodicity requirement $\Lambda(0) = \Lambda(T)$, $A(t)$ is brought into the time independent form $\tilde{A}$. This fixing still leaves the freedom of residual gauge transformation which are time independent. The Gauss constraint reduces the physical space of states to be invariant under this residual transformation. The wave function of the system depends on the gauge field $A$ and the fermionic degrees of freedom $\psi$. The gauge field $A$ transforms under the adjoint representation of the group. Therefore any function of $A$ transforms under a representation which is equivalent to the adjoint representation. We have seen that the anomaly condition implies that the Hilbert space of the fermions does not contain representations equivalent to the adjoint. In this case it is impossible to build a singlet from the fermionic degrees of freedom and functions of $A$. Therefore an anomalous system cannot satisfy Gauss' law.

The characterization given above for global anomalies in terms of different types of behaviour under a gauge transformation for the fields $\psi_i$ and for the states in the fermionic Hilbert space $\mathcal{S}$ is by no means special to zero-dimensional quantum mechanical systems. In $d + 1$ dimensions global anomalies may occur under space-time dependent gauge transformations which are in the non-trivial homotopy class $\pi_{d+1}(G)$, $G$ being the gauge group. The homotopy group $\pi_{d+1}(G)$ is the same as $\pi_1(g^d)$ where $g^d$ is the infinite-dimensional group of $d$-dimensional space-dependent gauge transformation. All the discussion following eq. (4.8) is true for the $d$-dimensional case with $r$ and $R$ being representations of the infinite-dimensional group $g^d$. The problem is again algebraic in nature, but involves the analysis of representations of infinite groups. Solutions were found in certain cases [2]. It seems again that for an anomalous theory it is impossible to couple the fermionic representation content with that of the gauge field into a singlet of $g^d$ as required by Gauss' law. The physical sector of the Hilbert space, which consists of gauge invariant states, turns out to be empty for a system with a global anomaly.

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References

    recent papers in ref.[4]