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PHASE STRUCTURE OF Z_p MODELS IN THE PRESENCE OF A θ PARAMETER*

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We investigate the phase structure of Z_p lattice gauge models in 4 dimensions in the presence of a θ-parameter. In this system Witten's result that monopoles acquire a fractional charge is readily obtained. A very rich phase structure is uncovered as a function of p, θ and the coupling constant. In addition to the usual electric and magnetic condensations, the phases, suggested by 't Hooft, of dyonic condensation and oblique confinement occur. A two-dimensional model with analogous structure is constructed and analyzed by renormalization group methods.

1. Introduction

Recently, 't Hooft [1] has argued that the relevant degrees of freedom of an SU(N) gauge theory on intermediate length scales may be realized by a partial fixing of the gauge, which leaves a maximal abelian subgroup $U(1)^{N-1}$ unbroken. The quarks and gluons of the theory have 'electric' charges with respect to these U(1) symmetries. The gauge condition becomes singular at points in three-dimensional space, which are magnetic monopoles with respect to the U(1) subgroups. Ordinary confinement occurs when these monopoles condense in the vacuum. In principle, other phases of the theory are possible: a Higgs phase where the electric charges condense and monopoles are confined, and a Coulomb phase with massless gauge bosons and no confinement.

However, this simple picture may change if the theory has a non-zero θ-angle. Indeed, it is already known from the case of the massive Schwinger [2] model that a θ like parameter may change the phase of the system. It was claimed by Callen, Dashen and Gross [3] that a similar phenomena will occur in $(QCD)_4$. 't Hooft [1] has shown that, due to the joint presence of gluons and monopoles in QCD, a rich phase structure may emerge as a function of θ. In that case, Witten [4] has argued

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that magnetic monopoles will also have electric charge, proportional to θ. 't Hooft then argues that, for $\theta \sim \pi$, monopoles are unlikely to condense at strong coupling, but instead bound states of monopoles and charges carrying small overall electric charge will condense. These phases have a number of unusual features, and are referred to as oblique confinement.

Because of the qualitative nature of these arguments, it is desirable to have a simple model in which these possibilities can be analysed in greater detail. We consider an abelian Z_P lattice gauge theory in four dimensions [5]. This model has excitations corresponding to Higgs particles, with charge p times the fundamental unit, which condense for weak coupling into a Higgs phase. These are also monopoles, resulting from the compact formulation of the lattice theory, which condense at strong coupling into a charge confining phase. For sufficiently large p, there is also an intermediate Coulomb phase, in which free photons exist. It is in this model that we now consider the effect of a non-zero θ angle. In the continuum limit, this corresponds to adding a term

$$\frac{i\theta g^2}{16\pi^2}\int F_{\mu\nu}\tilde{F}_{\mu\nu}\,\mathrm{d}^4x \tag{1.1}$$

to the action. In general, there is no really elegant way of writing the analogue of (1) for the lattice theory [6], but we find, in the abelian case, a relatively simple prescription which leads naturally to Witten's result on the dyonic nature of monopoles. Note that, although there is no necessity based on instanton ideas to add such a term to an abelian theory, it would play an important role in the presence of monopoles.

The phase structure of the model may be understood by estimating the free energies of the various possible condensates and choosing the lowest one. These estimates are somewhat crude, and to analyse their validity we consider an analogous two-dimensional spin model for which more precise statements based on the renormalization group can be made. The model is constructed from the gauge theory by a simple dimensional reduction procedure which ignores the dependence of the fields on two of the coordinates. The renormalization group analysis confirms the simple free energy arguments in this case, and the model turns out to be related to other interesting two-dimensional models. However, we do not pursue this question in this paper.

Our results can be summarized as follows. The phase diagram depends on p and θ. For a given θ, various phases are possible, and which ones are actually present depends to some extent on p. However, the generic phase diagram for $\theta \neq 0$ is shown in fig. 1. At weak coupling there is always a Higgs phase. This is followed by a Coulomb phase, an ordinary confining phase in which monopoles condense, a further Coulomb phase, and then a sequence of obliquely confining phases separated by Coulomb phases. For large p the ordinary confining phase is absent, while at

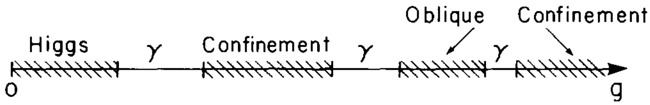

Fig. 1. Typical phase diagram. The number of oblique confinement phases depends on θ, and the Coulomb phases (γ) are absent for small p.

lower values of p the Coulomb phases may be absent. The number of obliquely confining phases depends dramatically on θ. If $\theta/2\pi$ has the form $1/q$ ($q=$ integer) there is only one such phase (and, in fact, the model is self-dual). If $\theta/2\pi = 2/q$ with q odd there are 2 such phases, and so on. If $\theta/2\pi$ is irrational there are in principle an infinite number of phase transitions.

In any given phase, particles whose electric and magnetic charges are the same multiple of those of a particle in the condensate, emerge as physical particles with only short-ranged forces between them. All other particles are confined by linear potentials. Thus, in an oblique confinement phase, both electric charges and magnetic monopoles, the basic excitations of the theory, are confined, but bound states of them may exist as free particles.

2. Definition of the model

The abelian gauge theory we consider is described by a partition function

$$Z = \mathrm{Tr}\exp\left[-\frac{1}{2g^2}\sum_{\mathrm{P}}\left(\Delta_\mu\phi_\nu - \Delta_\nu\phi_\mu - 2\pi S_{\mu\nu}\right)^2 + ip\sum_{\mathrm{L}} n_\mu\phi_\mu\right]. \tag{2.1}$$

The first term is a sum over plaquettes, where the ϕ_μ are variables defined on the links, and the $S_{\mu\nu}$ are integers defined on the plaquettes. Initially ϕ_μ is constrained to be in the interval $(-\pi, +\pi)$. However, because of the periodicity of (2.1) we can extend its domain to the real line. The link variables n_μ are integers defined on the links. When n_μ is traced over, the effect is to discretize ϕ_μ in units of $2\pi/p$. However, we can also add a term $\mu_c\Sigma_{\mathrm{L}} n_\mu^2$ to the action, which has the effect of controlling fluctuations in the n_μ. To maintain gauge invariance of (2.1) the n_μ must form closed loops, i.e. $\Delta_\mu n_\mu = 0$. We can think of these loops as the (euclidean) world-lines of the Higgs particles.

We have chosen the Villain form of the interaction because the monopoles can be identified explicitly. The monopole current is

$$m_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}\Delta_\nu S_{\lambda\sigma}. \tag{2.2}$$

A single monopole running, say, in the 4 direction can be represented by a configuration in which all the $S_{\lambda\sigma}$ on plaquettes dual to those on the world-sheet swept out by the Dirac string of the monopole are equal to unity, and all the other $S_{\lambda\sigma}$ are zero. Note that the m_μ lie on the links of the dual lattice. We can also add a term $\mu_{\mathrm{M}}\Sigma_{\tilde{\mathrm{L}}} m_\mu^2$, giving a bare monopole mass.

In the naive continuum limit, when monopoles are ignored, the rescaling $\phi_\mu = agA_\mu$ (where a is the lattice spacing) gives the standard lagrangian $\frac{1}{4}\int F^2 \mathrm{d}^4x$, and the Higgs particles are then seen to have charges pg.

The gauge field ϕ_μ may be completely integrated out of (2.1) to obtain a theory of interacting Higgs particles and monopoles:

$$Z = \mathrm{Tr}\exp\left[-\frac{2\pi^2}{g^2} \sum_{R,R'} m_\mu(R)G(R-R')m_\mu(R') - \tfrac{1}{2}p^2g^2 \sum_{r,r'} n_\mu(r)G(r-r')n_\mu(r') \right.$$
$$\left. + ip \sum_{R,r} m_\mu(R)n_\nu(r)\Theta_{\mu\nu}(R-r) \right]. \tag{2.3}$$

Here $G(R)$ is the 4-dimensional lattice Coulomb Green function. $\Theta_{\mu\nu}$ can be written as

$$\Theta_{\mu\nu}(R-r) = 2\pi\varepsilon_{\mu\nu\lambda\sigma}u_\lambda(u\cdot\Delta)^{-1}\Delta_\sigma^{(r)}G(R-r). \tag{2.4}$$

It is most easily visualized when, for example, m_μ represents a straight line in the 3-direction, and n_ν in the 4-direction. Then the final term in (2.3) depends only on the (1,2) coordinates of the charge and the monopole. It is, in fact, the angle which the vector $\boldsymbol{r}-\boldsymbol{R}$ makes with the Dirac string of the monopole, in the (1,2) plane. Thus, in general, $\Theta_{\mu\nu}$ changes by 2π whenever the electric charge crosses the world-sheet of the Dirac string of the monopole. This peculiar interaction is, in fact, rather elegant.

Suppose that in general we have two dyons, one at $\boldsymbol{R}$ with electric and magnetic charges (q, h), and one at $\boldsymbol{R}'$ with charges (q', h'). (These charges are pure numbers, expressed as multiples of the fundamental charges g and $2\pi/g$). According to (2.3), their contribution to the action is

$$i\left[hq'd_\mu(R)d_\nu(R')\Theta_{\mu\nu}(R-R') + h'qd_\mu(R')d_\nu(R)\Theta_{\mu\nu}(R'-R)\right], \tag{2.5}$$

where d_μ is the dyon current. Now $\Theta_{\mu\nu}(R-R')$ is symmetric in $(R-R')$, but is antisymmetric in μ and ν. Thus (2.5) may be written

$$i(hq'-h'q)d_\mu(R)d_\nu(R')\Theta_{\mu\nu}(R-R'). \tag{2.6}$$

However, $\Theta_{\mu\nu}$ is an angle, and so the partition function should not change when $\Theta_{\mu\nu}$ is increased by 2π. So

$$hq'-h'q = \text{integer}. \tag{2.7}$$

In our units, this is the generalization [7] of the Dirac [8] quantization condition to particles carrying electric and magnetic charge. It is built into the lattice theory. Of course, the excitations in our model (2.1) automatically satisfy this criterion. It was pointed out by Witten [4] that (2.7) allows monopoles to carry non-integer electric charge proportional to their magnetic charge. This actually occurs when there is a non-zero θ angle.

In order to discuss the effect of the θ-angle, we first consider the continuum limit

$$\int F_{\mu\nu}\tilde{F}_{\mu\nu}\mathrm{d}^4x = \int \frac{\partial}{\partial x_\mu}\left(\varepsilon_{\mu\nu\lambda\sigma}A_\nu \frac{\partial A_\lambda}{\partial x_\sigma}\right)\mathrm{d}^4x. \tag{2.8}$$

Suppose we have a static monopole at the origin:

$$m_\mu = m\delta_{\mu 4}\delta_{\boldsymbol{R},0}. \tag{2.9}$$

Consider the contribution to (2.8) from a hypercube containing the origin. The field A_4 will be smooth over the interior of the cube, but the space components A_1, A_2, A_3 will be rapidly varying in space, but not imaginary time. Thus the term in (2.8) with $\nu = 4$ can be approximated by A_4:

$$A_4\int \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk}\frac{\partial A_j}{\partial x_k}\right) = A_4\int_S(\nabla \times \boldsymbol{A})\cdot \mathrm{d}\boldsymbol{S}, \tag{2.10}$$

where the first integral is over the volume, and the second over the surface of a 3-cube containing the origin. The second integral is of course just the flux of the $\boldsymbol{B}$-field out of the cube, so it is proportional to m.

Thus we see that the effect of the term $F\tilde{F}$ is simply to couple A_μ to the monopole current m_μ. There is a problem in defining this on the lattice however, since the fields $\phi_\mu \propto A_\mu$, and the monopole current m_μ are defined on different lattices, one being the dual of the other.

We therefore include the effect of the θ-angle by adding to the action a term

$$\frac{ip\theta}{2\pi}\sum_{r,R} f(r-R)\phi_\mu(r)m_\mu(R), \tag{2.11}$$

where $f(r-R)$ is short-ranged, and $\Sigma_R f(r-R) = 1$. The exact form of f should not affect the large-distance physics. Comparing with (2.1) we see that the effect of this term is to make m_μ behave as if it carried an electric current $(\theta/2\pi)m_\mu$. The fields ϕ_μ

may once again be integrated out, to obtain, instead of (2.3),

$$Z = \mathrm{Tr}\exp\left[-\frac{2\pi^2}{g^2}\sum_{R,R'} m_\mu(R)G(R-R')m_\mu(R') - \tfrac{1}{2}p^2g^2\sum_{r,r'}\sum_{R,R'}\left(n_\mu(r)+\frac{\theta}{2\pi}f(r-R)m_\mu(R)\right) \times G(r-r')\left(n_\mu(r')+\frac{\theta}{2\pi}f(r'-R')m_\mu(R')\right) + ip\sum_{R,r} m_\mu(R)n_\nu(r)\Theta_{\mu\nu}(R-r)\right]. \quad (2.12)$$

Note that θ drops out of the last term, so that the partition function is still single valued. This is consistent with the observation of Witten [4] that a non-zero θ angle gives charges consistent with the generalized Dirac condition.

In what follows we shall replace f by a delta function and ignore the fact that n_μ and m_μ lie on different lattices. This should not affect the large-distance physics. The partition function is then seen to be a periodic function of θ, since $\theta \to \theta + 2\pi$ may be compensated by a change $n_\mu \to n_\mu - m_\mu$. Note that this is no longer true if we add mass terms to (2.12). In a non-abelian theory, we would not have to put in Higgs particles, since the gluons themselves carry charge. Because of the periodicity we can restrict attention to the interval $0 \leq \theta \leq \pi$. Under the parity operation, $n_\mu \to -n_\mu$ but $m_\mu \to m_\mu$. So the action is invariant under parity only at $\theta = 0$ or π.

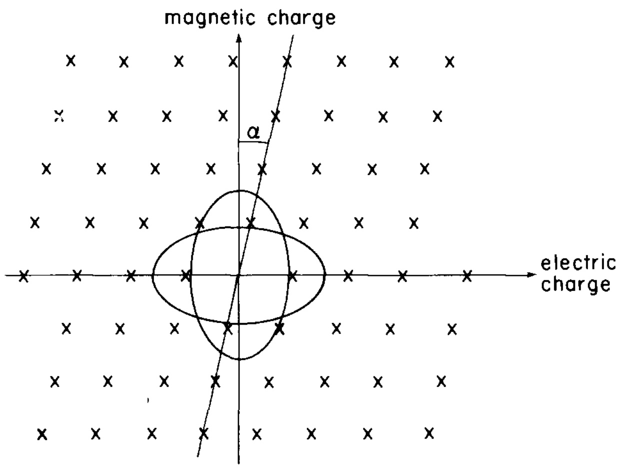

Fig. 2. The charge lattice. The angle $\alpha = \tan^{-1}(\theta/2\pi)$. Two ellipses for different values of g are shown. Charges within the ellipse can condense.

The electric and magnetic charges of the excitations of the model may be plotted in a plane (fig. 2). They form a lattice, which is not rectilinear unless θ is a multiple of 2π.

3. Analysis of the phase diagram

3.1. DUALITY

When $\theta = 0$, it is well known that this model is self-dual [5]. This is apparent in (2.12). Under duality,

$$g^2 \to 4\pi^2/p^2g^2. \tag{3.1}$$

It turns out that the model is also self-dual when $2\pi/\theta$ is an integer, although in a different way. Suppose that $\theta/2\pi = 1/q$. The action may be written symbolically as

$$\frac{2\pi^2}{g^2} m_\mu G m_\mu + \tfrac{1}{2}p^2g^2\left(n + \frac{m}{q}\right)_\mu G\left(n + \frac{m}{q}\right)_\mu + ipm_\mu \Theta_{\mu\nu} n_\nu . \tag{3.2}$$

Let

$$n_\mu + \frac{m_\mu}{q} = \frac{-l_\mu}{q}, \tag{3.3}$$

where l_μ is an integer. We may label the points of the charge lattice by (n, l) instead of (n, m). Note that this does not work for arbitrary rational $\theta/2\pi$. (3.2) may then be written

$$\frac{2\pi^2}{g^2} q^2\left(n + \frac{l}{q}\right)_\mu G\left(n + \frac{l}{q}\right)_\mu + \frac{p^2g^2}{2q^2} l_\mu G l_\mu - ipl_\mu \theta_{\mu\nu} n_\nu . \tag{3.4}$$

The change of sign in the last term is irrelevant. (3.4) is equivalent to (3.2) with

$$g^2 \to 4\pi^2q^2/p^2g^2. \tag{3.5}$$

The self-dual value of g^2 is $2\pi q/p$, and the phase diagram must be inversion symmetric about this point. Note that we get the conventional result if we take $q = 1$ and use periodicity in θ.

3.2. FREE ENERGY ARGUMENTS

In the U(1) theory, simple arguments have been used [9] to show that monopoles should condense at strong coupling. It is straightforward to extend to this problem. Consider a large loop of length L, carrying electric and magnetic charge $(n +$

$(\theta/2\pi)m, m)$. We can estimate its energy by taking only the terms in the action with $r=r'$. This gives

$$\left(\frac{2\pi^2}{g^2}m^2+\tfrac{1}{2}p^2g^2\left(n+\frac{\theta m}{2\pi}\right)^2\right)G(0)L. \tag{3.6}$$

On the other hand, its entropy is also proportional to L, roughly $L\ln 7$, since at each step the loop can choose seven different directions. Thus long, tangled loops should condense in the vacuum if

$$\left(\frac{m^2}{T}+\left(n+\frac{\theta}{2\pi}m\right)^2T\right)p<C, \tag{3.7}$$

where $T=pg^2/2\pi$, and $C=(\ln 7)/\pi G(0)$. The value of C is, of course, not to be taken too seriously. We have neglected the long-range part of the force completely, assuming that it is screened by other loops. Note, however, that the phase diagram will depend on C/p, so lack of knowledge of C only affects the critical values of p. When $\theta=0$, the criterion (3.7) suggests the existence of an intermediate Coulomb phase, where neither Higgs particles nor monopoles can condense, when $p>C$. Since such a phase is observed in Monte Carlo simulations [10] for $p\geqslant 6$, we can suppose that C is in this region. The two-dimensional model to be discussed later gives $C=4$.

When two or more condensates are possible, we choose the one with the lowest free energy, that is the smallest value of the left-hand side of (3.7). In a given direction away from the origin of the charge lattice, we have to consider only the points closest to the origin. It is not possible for two condensations (m,n) and (m',n') to happen simultaneously if $m/n\neq m'/n'$. If this were the case, the free energy, by (2.11), would contain a term

$$ip(mn'-m'n)\sum_{R,r}\Theta_{\mu\nu}(R-r). \tag{3.8}$$

Since $\Theta_{\mu\nu}$ is an angle defined with respect to an arbitrary direction, the partition function would not be well-defined, or the theory would not be covariant at large distances.

The criterion (3.7) defines the interior of an ellipse in the plane of the charge lattice. The ratio of its axes is T, and its area is $\pi C/p$. For p large, there will be no points of the charge lattice inside the ellipse besides the origin, unless T is either very large or very small. Thus at intermediate coupling, there will be a Coulomb phase. For small p, there will always be points inside the ellipse, and no Coulomb phase. These are the general features. We now consider some particular values of θ.

(a) $\theta=\pi$. The charge lattice is shown in fig. 3. Only the points labelled H (the elementary Higgs particle), M (the elementary monopole) and O (a composite with $n=-1$, $m=2$), together with their charge conjugates, are possible candidates for condensates. The phase diagram for different values of p/C is illustrated in fig. 4. The strong coupling phase is where O condenses. This is the example of oblique confinement quoted by 't Hooft.

(b) $\theta/2\pi = 1/N$. The qualitative picture does not change from case (a), only the various critical values of p/C. The phase diagram remains self-dual.

(c) $\theta/2\pi = 2/N$, N odd. As an example we choose $N = 5$. The charge lattice is shown in fig. 5. There are now two points O_1 and O_2 which may condense at different couplings, and give the possibility of two different obliquely confining phases. The phase diagram as we vary p is shown in fig. 6.

Clearly one can investigate more complicated situations. If $\theta/2\pi$ is rational, there will always be a point of the charge lattice with zero electric charge, as this will always condense at sufficiently strong coupling, although there may be many different oblique confining phases until this limit is reached. If $\theta/2\pi$ is irrational, however, all states will have non-zero electric charge and thus will ultimately pass out of the ellipse. Thus in our approximation there will be an infinite sequence of obliquely confining phases as we pass to higher and higher values of g^2. The discontinuous behavior in terms of θ may raise concern in an attempt to renormalize the parameter θ. We comment that by adding a term $\mu^2 \Sigma_i n^2(i)$ to the action (2.12) one describes the Villain version of an abelian U(1) matter field coupled to a U(1) gauge theory in the presence of a θ parameter. This mass term would suppress the infinity of phase transitions obtained for the Z_p theory.

In order to understand the properties of all these phases, we need to calculate Wilson loops. This calculation is carried out in the appendix. In a phase in which the

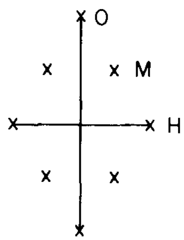

Fig. 3. Charge lattice for $\theta = \pi$.

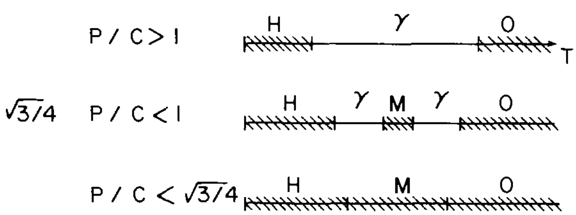

Fig. 4. Phase diagrams for $\theta = \pi$.

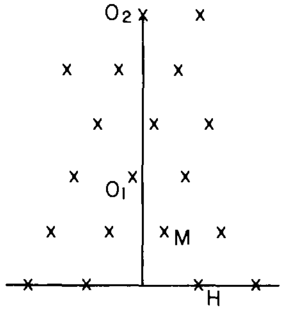

Fig. 5. Charge lattice for $\theta = \frac{4}{5}\pi$.

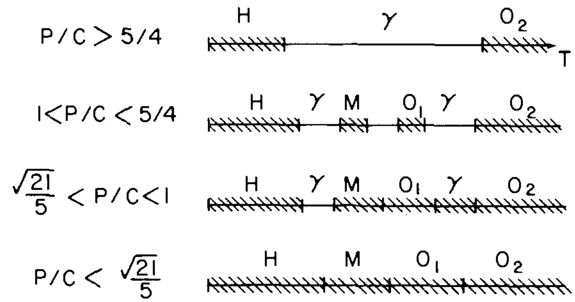

Fig. 6. Phase diagram for $\theta = \frac{4}{5}\pi$.

charge (n, m) condenses all particles which have quantum numbers (n', m') not in the condensate have Wilson loops decaying with an area law, so are confined. The string tension is proportional to $(mn' - m'n)^2$, the 'distance' of the particle away from the condensate on the charge lattice. Particles in the condensate have Wilson loops decaying with a perimeter law and so are unconfined. In fact they interact via short-ranged forces, whose inverse range is characterized by a finite mass scale.

4. Two-dimensional model

The above results depend rather crucially on the form of the left-hand side of (3.7). We now consider a related two-dimensional model where the free energy arguments can be backed up by a renormalization group analysis. We construct the model from the 4D gauge theory by assuming that ϕ_1, ϕ_2 vanish and ϕ_3, ϕ_4 depend only on x_1, x_2. The partition function for one slice $x_3, x_4 = $ constant in the absence of the θ-angle is

$$Z = \mathrm{Tr}\exp\left[\frac{1}{2g^2}\sum_{\mu, a}(\Delta_\mu\phi_a - 2\pi S_{\mu a})^2 + ip\sum_a n_a\phi_a\right], \qquad (4.1)$$

where $\mu = 1,2$ and $a = 3,4$. The fields ϕ_a may now be thought of as residing on the sites of the 2 D lattice, and the integer variables $S_{\mu a}$ on the links. The (3,4) labels have become internal symmetry indices, as is common in dimensional reduction.

The model (4.1) corresponds to two decoupled Z_p spin models, or clock models. The non-zero monopole currents

$$m_3 = -\Delta_1 S_{24} + \Delta_2 S_{14}, \tag{4.2}$$

$$m_4 = \Delta_1 S_{23} - \Delta_2 S_{13}, \tag{4.3}$$

now represent vorticity. The vorticity of the ϕ_3 field is m_4, that of the ϕ_4 field is $-m_3$. The electric currents n_3, n_4 correspond to what are sometimes called spin wave excitations of the clock models.

The phase diagram of the Z_p clock model is well understood [5,11]. For $p<4$ there is a low temperature (weak coupling) phase where the spin wave operators gain an expectation value (they 'condense'). Unlike the gauge theory, the Z_p symmetry is global and is spontaneously broken in this phase. Vortices are bound by a linear potential; that is, their correlations decay exponentially. The high temperature dual of this phase is one where the vortex interactions are screened and spin-spin correlations decay exponentially. For $p>4$, there is an intermediate massless phase. Thus this model has may similarities with the 4D gauge theory.

The fields ϕ_3, ϕ_4 may be integrated out of (4.1) to obtain [11]

$$\begin{aligned} Z = \operatorname{Tr} \exp \Bigg[& \frac{\pi}{g^2} \sum_{R \neq R'} \left(m_3(R) m_3(R') + m_4(R) m_4(R') \right) \ln(R - R') \\ & + \frac{p^2 g^2}{4\pi} \sum_{r \neq r'} \left(n_3(r) n_3(r') + n_4(r) n_4(r') \right) \ln(r - r') \\ & + ip \sum_{r, R} \left(m_3(R) n_4(r) - m_4(R) n_3(r) \right) \Theta(r - R) \\ & - \tfrac{1}{8} p^2 g^2 \sum_r \left(n_3^2(r) + n_4^2(r) \right) - \frac{\pi^2}{2g^2} \sum_R \left(m_3^2(R) + m_4^2(R) \right) \Bigg], \end{aligned} \tag{4.4}$$

where we have used the asymptotic form $-(1/2\pi)\ln r$ for the 2D Coulomb Green function. The function $\Theta(r-R)$ is the angle which the vector $(\boldsymbol{r}-\boldsymbol{R})$ makes with a fixed direction.

When we add the θ-angle, it is actually possible to avoid the introduction of the arbitrary function $f(R-r)$ of (2.11), if we first modify the model so that ϕ_4 is defined on the sites $\boldsymbol{R}$ of the dual lattice. Since m_4 is the vorticity for ϕ_3, it is also defined on the dual lattice, so there is now no difficulty in writing down a purely local interaction

$$\frac{ip\theta}{2\pi}\Big(\sum_r m_3(r)\phi_3(r)+\sum_R m_4(R)\phi_4(R)\Big). \tag{4.5}$$

Comparing with (4.1), we see that vortices of ϕ_3 couple as spin-wave excitations to ϕ_4, and vice versa. The Coulomb gas representation is now

$$\begin{aligned} Z=\operatorname{Tr}\exp\Bigg[&\frac{\pi}{g^2}\sum_{r\neq r'} m_3(r)m_3(r')\ln(r-r')+\frac{\pi}{g^2}\sum_{R\neq R'} m_4(R)m_4(R')\ln(R-R') \\ &+\frac{p^2g^2}{4\pi}\sum_{r\neq r'}\Big(n_3(r)+\frac{\theta}{2\pi}m_3(r)\Big)\Big(n_3(r')+\frac{\theta}{2\pi}m_3(r')\Big)\ln(r-r') \\ &+\frac{p^2g^2}{4\pi}\sum_{R\neq R'}\Big(n_4(R)+\frac{\theta}{2\pi}m_4(R)\Big)\Big(n_4(R')+\frac{\theta}{2\pi}m_4(R')\Big)\ln(R-R') \\ &+ip\sum_{r,R}\big(m_3(r)n_4(R)-m_4(R)n_3(r)\big)\theta(r-R)\Bigg] \\ &-\tfrac{1}{8}p^2g^2\Big(\sum_r n_3^2(r)-\sum_R n_4^2(R)\Big)-\frac{\pi^2}{2g^2}\Big(\sum_r m_3^2(r)+\sum_R m_4^2(R)\Big). \end{aligned} \tag{4.6}$$

For the energy to be finite, we impose the conditions $\Sigma m_3=\Sigma m_4=\Sigma n_3=\Sigma n_4=0$.

Note that if we had tried to add a θ-angle term to a single clock model, the final angular term would depend upon θ, and the partition function would not be single valued. We need the extra antisymmetry provided by the internal symmetry.

The phase diagram of this model may be analysed in two ways. First, one can make simple free energy arguments following Kosterlitz and Thouless [12]. If the system has linear size L, the energy of a single excitation with quantum numbers $(n_3=n,\ m_3=m,\ n_4=m_4=0)$ is

$$\left[\frac{\pi^2}{2g^2}m^2+\tfrac{1}{8}p^2g^2\Big(n+\frac{\theta}{2\pi}m\Big)^2\right]\ln L, \tag{4.7}$$

while its entropy is $\sim\ln L^2$. The condensation of such objects will therefore be favored when

$$\frac{\pi^2}{2g^2}m^2+\tfrac{1}{8}p^2g^2\Big(n+\frac{\theta}{2\pi}m\Big)^2<2, \tag{4.8}$$

a result which is identical to (3.7) with $C=4$.

Alternatively one may use a renormalization group argument to study the stability of the gaussian phase, following Jose et al. [9]. First one adds to the action of (3.5) a term

$$\sum_{r}\sum_{n',m'}(\ln y_{n',m'})\delta_{n_3(r),n'}\delta_{m_3(r),m'}+\sum_{R}\sum_{n'm'}(\ln y_{n',m'})\delta_{n_4(R),n'}\delta_{m_4(R),m'}. \quad (4.9)$$

The new variables $y_{n,m}$ act as fugacities for charges (n,m), and serve to count them in the partition function.

Because of total charge neutrality, the first terms in the power series development of Z in the $y_{n,m}$ are

$$Z=1+2\sum_{n,m}y_{n,m}^2\int\frac{d^2r}{a^2}\frac{d^2r'}{a^2}\left(\frac{r-r'}{a}\right)^{-(2\pi/g^2)m^2-(p^2g^2/2\pi)(n+\theta m/2\pi)^2}+\cdots, \quad (4.10)$$

where we have replaced sums by integrals, and been careful to introduce the lattice spacing a to be dimensionally correct. Under a rescaling $a\to ba$, $y_{n,m}$ must rescale according to

$$y_{n,m}\to y_{n,m}b^{(2-\pi m^2/g^2+(p^2g^2/4\pi)(n+\theta m/2\pi)^2)} \quad (4.11)$$

so that Z be unchanged. Thus, when the inequality (4.8) is satisfied, $y_{n,m}$ is relevant on the gaussian fixed line $y_{n,m}=0$. Of course, this is only a local statement valid near the gaussian line, and we must assume that the flows are uninterrupted as we continue to $y_{n,m}=1$. It is possible to write down the full renormalization group equations to $O(y^2)$, but they are not very instructive. The main point is that the form of the criterion (4.8) is substantiated in this case by renormalization group arguments. Thus we expect this two-dimensional model to have the rich phase structure as indicated in the previous section for the gauge theory. It would be interesting to study these models further. One special case ($p=2, \theta=\pi$) corresponds to the Ising representation of the F-model. Our results agree with the exact solution for this case [13].

5. Conclusions

We have constructed abelian systems with a local gauge symmetry in four dimensions and a global symmetry in two dimensions. The phase structure in these models is enriched by adding a θ-like parameter. In our formulation one easily detects that monopoles pick up a fractional electric charge in the presence of a θ parameter as first pointed out by Witten. As a consequence we demonstrated using renormalization group arguments that dyon condensation as well as oblique confinement does indeed occur in 2 dimensions. Applying cruder arguments we show that the phases suggested by 't Hooft for SU(N) gauge theories occurs also in 4-dimensions in the model discussed. The systems we study are all abelian, nevertheless they are also boundaries of non-abelian systems coupled to Higgs fields. Thus the structure uncovered will also appear in non-abelian theories.

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Appendix

The Wilson loop for a particle with quantum numbers (n', m') may be evaluated by replacing

$$n_\mu(r) \to n_\mu(r) + N_\mu(r), \qquad m_\mu(R) \to m_\mu(R) + M_\mu(R) \tag{A.1}$$

in (2.12), where $(N_\mu(r), M_\mu(r)) = (n', m')$ on a link of the loop, and zero elsewhere. If we call this modified partition function $Z\{N_\mu, M_\mu\}$, the Wilson loop is

$$W(C) = Z\{N_\mu, M_\mu\}/Z\{0,0\}. \tag{A.2}$$

In what follows, we introduce the electric charge current $q_\mu = n_\mu + (\theta/2\pi)m_\mu$, and similarly define $Q_\mu = N_\mu + (\theta/2\pi)M_\mu$. Suppose now that the charge (q, m) condenses, where $q/m = \tan\alpha$. The value of (n, m) in the condensate will dominate the partition sum. Writing for those values $q = \xi\sin\alpha$, $m = \xi\cos\alpha$, the sum over the allowed values of q, m may be replaced by an integral over ξ, if we use the Poisson sum formula:

$$\sum_{n,m} = \sum_{s,t} \exp\left[2\pi i\xi\left(s\left(\sin\alpha - \frac{\theta}{2\pi}\cos\alpha\right) + t\right)\right]\int_{-\infty}^{\infty} d\xi\, e^{-\lambda\xi^2}. \tag{A.3}$$

We have introduced by weight factor $e^{-\lambda\xi^2}$ because we expect, under renormalization, a given particle to gain a self-energy proportional to the square of its charge. We now consider only the term with $s = t = 0$. A similar calculation has been carried out by Pelcovits [14] for the spin-spin correlation function in the high temperature phase of the XY model. In that case, terms we neglect correspond to fluctuations of the string connecting the two spins. They lead to the prefactor in his result $R^{-1/2}e^{-\mu R}$. In our case, non-zero s, t correspond to fluctuations of the Wilson surface, and lead to a non-leading Coulomb term in the potential [15].

With these approximations Z takes the form

$$Z\{N_\mu, M_\mu\} = \int \prod_r \mathrm{d}\xi_\mu(r) \mathrm{e}^{-\lambda\xi_\mu(r)^2}$$

$$\times \exp\Big[-\frac{2\pi^2 2}{g^2} \sum_{r,r'} \big(\xi_\mu(r)\cos\alpha + M_\mu(r)\big)$$

$$\times G(r-r')\big(\xi_\mu(r')\cos\alpha + M_\mu(r')\big)$$

$$-\tfrac{1}{2}p^2g^2 \sum_{r,r'} \big(\xi_\mu(r)\sin\alpha + Q_\mu(r)\big) G(r-r')\big(\xi_\mu(r')\sin\alpha + Q_\mu(r')\big)$$

$$+ip \sum_{r,r'} \big(\xi_\mu(r)\cos\alpha + M_\mu(r)\big)\big(\xi_\nu(r')\sin\alpha + Q_\nu(r')\big)\theta_{\mu\nu}(r-r')\Big]. \tag{A.4}$$

This is a gaussian integral over the $\xi_\mu(r)$. Write

$$M_\mu(r) = m'J_\mu(r), \qquad Q_\mu(r) = q'J_\mu(r). \tag{A.5}$$

The terms in (A.4) which do not involve $\xi_\mu(r)$ are Coulomb interactions between the $J_\mu(4)$. They lead to a perimeter dependence. The integration over the $\xi_\mu(r)$ leads to a result which has the symbolic form, in momentum space,

$$\exp\left[J_\mu \frac{1}{k^2} \cdot \left(\frac{A}{k^2} + \lambda\right)^{-1} \frac{1}{k^2} J_\mu - p^2(q'\cos\alpha - m'\sin\alpha)^2 J_\mu \theta_{\mu\nu} \left(\frac{A}{k^2} + \lambda\right)^{-1} \theta_{\nu\lambda} J_\lambda \right], \tag{A.6}$$

where

$$A = \frac{2\pi^2}{g^2}\cos^2\alpha + \tfrac{1}{2}p^2g^2\sin^2\alpha. \tag{A.7}$$

We have used the fact that $G \sim 1/k^2$ in momentum space. The first term in (A.6) behaves like $1/k^2$ as $k \to 0$, giving again a perimeter law. The second term, however, is proportional to the area of the loop. To see this, suppose J_μ represent static charges a distance L apart:

$$J_\mu(r) = \delta_{\mu 4}\big[\delta(x_1) - \delta(x_1 - L)\big]\delta(x_2)\delta(x_3). \tag{A.8}$$

We then get a contribution in (A.6) with $\nu=2,3$. The Fourier transform of $(A/k^2+\lambda)^{-1}$ is short-ranged. For $L \gg (\lambda/A)^{1/2}$ we may replace it by a nearest-neighbor interaction. Substituting in (A.6) we obtain for $\nu=2$

$$p^2(q'\cos\alpha - m'\sin\alpha)^2 \sum_{\langle \boldsymbol{R},\boldsymbol{R}\rangle} (u(R)-u(R'))^2. \tag{A.9}$$

Here $u(R)$ is the angular variable following eq. (2.4). $\boldsymbol{R}$ and $\boldsymbol{R}'$ lie in the (1,3) plane. $u(\boldsymbol{R})$ has a discontinuity of 2π across the line joining the two sources. Thus (A.9) is proportional to the length of the string joining them. The string tension is proportional to $(q'\cos\alpha - m'\sin\alpha)^2$, so it vanishes when $q'/m' = \tan\alpha$, i.e. the charge lies in the condensate. In that case we must consider all the terms in (A.4). Writing $Q_\mu = J_\mu(r)\sin\alpha$, $M_\mu = J_\mu(r)\cos\alpha$, (A.4) simplifies to

$$\int \prod_r \mathrm{d}\xi_\mu(r) \mathrm{e}^{-\lambda\xi_\mu(r)^2} \exp\left[-A(\xi_\mu + J_\mu)\frac{1}{k^2}(\xi_\mu + J_\mu)\right]. \tag{A.10}$$

The integration is straightforward, and leads to

$$\exp\left[-J_\mu \frac{A\lambda}{\lambda k^2 + A} J_\mu\right]. \tag{A.11}$$

This gives a perimeter law. The potential between two charges in the condensate is e^{-Mr} where $M = (A/\lambda)^{1/2}$, a short-range force.

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